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Finite generation of rings of differential operators of semigroup algebras

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Abstract

We prove that the ring of differential operators of any semigroup algebra is finitely generated. In contrast, we also show that the graded ring of the order filtration on the ring of differential operators of a semigroup algebra is finitely generated if and only if the semigroup is scored.

Keywords: Toric variety; Semigroup ring; Finite generation; Ring of differential operators

1. Introduction

This paper investigates conditions under which various rings of differential operators on semigroup algebras are finitely generated. The ring of differential operators $D(R)$ on the algebra R was introduced by Grothendieck [4] and Sweedler [15]. Many recent papers describe the structure of the ring of differential operators for special classes of algebras. For instance, Jones [6], Musson [8], and Musson and Van den Bergh [9] characterize $D(R)$ when R is the coordinate ring of a normal toric variety. In this case $D(R)$ inherits a fine grading from R, and both $D(R)$ and $\mathrm{Gr}(D(R))$—the graded ring of differential operators with respect to the order filtration—are finitely generated algebras (see [7,14] for other approaches to this result).

Given a finite set A of integral vectors and a parameter vector β, Gel'fand, Kapranov, and Zelevinskii defined and studied a system of differential equations, the A-hypergeo-

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metric system $H_A(\beta)$ ([2,3], etc.; also see [11]). The symmetry algebra of the systems—the algebra of contiguity operators—controls homomorphisms between systems with different parameter vectors [10]. We showed in a previous paper [12] that the symmetry algebra is anti-isomorphic to the ring of differential operators $D(R_A)$ for the semigroup algebra $R_A = \mathbb{C}[\mathbb{N}A]$. This connection to A-hypergeometric systems motivates our study of differential operators on semigroup algebras but we feel that the ring $D(R_A)$ is also interesting in its own right.

While considering the finite generation of $\mathrm{Gr}(D(R_A))$ in our previous paper [12], we defined the notion of a scored semigroup: a semigroup $\mathbb{N}A$ is scored if the difference $(\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}A) \setminus \mathbb{N}A$ consists of a finite union of hyperplane sections of $\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}A$ parallel to facets of the cone $\mathbb{R}_{\geqslant 0}A$. We conjectured the following in [12] and prove it in this paper.

Theorem 1.1 [12, Conjecture 3.2.10]. *Let $R_A = \mathbb{C}[\mathbb{N}A]$ be a semigroup algebra. Then*:

(1) $\mathrm{Gr}(D(R_A))$ *is finitely generated* $\Leftrightarrow$ $\mathbb{N}A$ *is a scored semigroup.*
(2) $D(R_A)$ *is finitely generated for all semigroup algebras* R_A.

Earlier we proved the $\Rightarrow$ direction of (1) [12, Theorem 3.2.12]. We also proved the $\Leftarrow$ direction of (1) when the cone $\mathbb{R}_{\geqslant 0}A$ is generated by linearly independent vectors [12, Theorem 3.2.13].

The layout of this paper is as follows: we start by reviewing some fundamental facts about the ring $D(R_A)$ and introducing some notation in Section 2. At some point there was confusion in the research community about the relationship between scored and Cohen–Macaulay semigroup algebras. In Section 2 we provide two examples to show that neither condition implies the other. We describe the difference $(\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}A) \setminus \mathbb{N}A$ in terms of associated primes in Section 3. We use this description in Section 4 to decompose the lattice $\mathbb{Z}A$ into finite pieces suitable for the arguments in Sections 5 and 6. A running example is used to illustrate our definitions. In Section 5 we prove that the ring of differential operators $D(R_A)$ is finitely generated for any semigroup algebra R_A (Theorem 1.1(2)). Example 5.3 is intended to orient the reader to the structure of our argument while illustrating our approach to proving the finite generation of $D(R_A)$. In the final section we complete the proof of Theorem 1.1(1) by showing that the graded ring of differential operators $\mathrm{Gr}(D(R_A))$ is finitely generated if R_A is a scored semigroup algebra. We also show that $D(R_A)$ is left and right Noetherian when $\mathbb{N}A$ is a scored semigroup.

2. Rings of differential operators of semigroup algebras

In this section, we briefly recall some fundamental facts about the rings of differential operators of semigroup algebras. Let

$$A := \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \tag{1}$$

be a finite set of integral vectors in $\mathbb{Z}^d$. Sometimes we identify A with the matrix of column vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$. *Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$, for simplicity.*

The ring of differential operators with Laurent polynomial coefficients

$$D\bigl(\mathbb{C}\bigl[\mathbb{Z}^d\bigr]\bigr) := \mathbb{C}\bigl[t_1^{\pm 1}, \ldots, t_d^{\pm 1}\bigr]\langle \partial_1, \ldots, \partial_d \rangle$$

is the ring of differential operators on the algebraic torus $(\mathbb{C}^\times)^d$, where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$, and the other pairs of generators commute. Here $[\,,]$ denotes the commutator and δ_{ij} is 1 if $i = j$ and 0 otherwise.

2.1. *The rings R_A and $D(R_A)$*

The semigroup algebra $R_A := \mathbb{C}[\mathbb{N}A] = \bigoplus_{\mathbf{a} \in \mathbb{N}A} \mathbb{C} t^{\mathbf{a}}$ is the ring of regular functions on the affine toric variety defined by A, where $t^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\mathbf{a} = {}^t(a_1, a_2, \ldots, a_d)$, the transpose of the row vector $(a_1, \ldots, a_d)$. Its ring of differential operators $D(R_A)$ can be realized as a subring of the ring $D(\mathbb{C}[\mathbb{Z}^d])$ of differential operators on the big torus as follows:

$$D(R_A) = \bigl\{P \in D\bigl(\mathbb{C}\bigl[\mathbb{Z}^d\bigr]\bigr):\ P(R_A) \subset R_A\bigr\}.$$

Put $\theta_j := t_j \partial_j$ for $j = 1, 2, \ldots, d$. Then it is easy to see that $\theta_j \in D(R_A)$ for all j. We introduce a grading on the ring $D(R_A)$ as follows: for $\mathbf{a} = {}^t(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$, set

$$D(R_A)_{\mathbf{a}} := \bigl\{P \in D(R_A):\ [\theta_j, P] = a_j P \text{ for } j = 1, 2, \ldots, d\bigr\}.$$

Then $D(R_A)$ is $\mathbb{Z}^d$-graded: $D(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} D(R_A)_{\mathbf{a}}$.

We introduce some notation to describe the graded structure of the ring of differential operators $D(R_A)$ explicitly. For $\mathbf{d} \in \mathbb{Z}^d$, we define a subset $\Omega(\mathbf{d})$ of the semigroup $\mathbb{N}A$ by

$$\Omega(\mathbf{d}) = \{\mathbf{a} \in \mathbb{N}A:\ \mathbf{a} + \mathbf{d} \notin \mathbb{N}A\} = \mathbb{N}A \setminus (-\mathbf{d} + \mathbb{N}A).$$

Theorem 2.1 ([6], [12, Theorem 3.3.1]).

$$D(R_A) = \bigoplus_{\mathbf{d} \in \mathbb{Z}^d} D(R_A)_{\mathbf{d}} = \bigoplus_{\mathbf{d} \in \mathbb{Z}^d} t^{\mathbf{d}} \mathbb{I}\bigl(\Omega(\mathbf{d})\bigr),$$

where

$$\mathbb{I}\bigl(\Omega(\mathbf{d})\bigr) := \bigl\{f(\theta) \in \mathbb{C}[\theta] := \mathbb{C}[\theta_1, \ldots, \theta_d]:\ f \text{ vanishes on } \Omega(\mathbf{d})\bigr\}.$$

In [12], we conjectured that $D(R_A)$ is finitely generated for all semigroup algebras R_A.

2.2. *The ring* $\mathrm{Gr}(D(R_A))$

Next we explain the order filtration. A differential operator

$$P = \sum_{\mathbf{a} \in \mathbb{N}^d} a_{\mathbf{a}}(t) \partial^{\mathbf{a}} \in D\bigl(\mathbb{C}\bigl[\mathbb{Z}^d\bigr]\bigr)$$

is said to be of *order* k if $a_{\mathbf{a}} \neq 0$ for some $\mathbf{a}$ with $|\mathbf{a}| = k$ and $a_{\mathbf{a}} = 0$ for all $\mathbf{a}$ with $|\mathbf{a}| > k$, where $|\mathbf{a}| = a_1 + a_2 + \cdots + a_d$. Let $D_k(R_A)$ denote the set of differential operators in $D(R_A)$ of order at most k. Then $\{D_k(R_A)\}_{k\in\mathbb{N}}$ is called the *order filtration* of $D(R_A)$. We consider the graded ring $\mathrm{Gr}(D(R_A))$ of $D(R_A)$ with respect to the order filtration:

$$\mathrm{Gr}\bigl(D(R_A)\bigr) := \bigoplus_{k\in\mathbb{N}} D_k(R_A)/D_{k-1}(R_A),$$

where we put $D_{-1}(R_A) = 0$. The graded ring $\mathrm{Gr}(D(R_A))$ is a subring of the commutative ring

$$\mathrm{Gr}\bigl(D\bigl(\mathbb{C}\bigl[\mathbb{Z}^d\bigr]\bigr)\bigr) = \mathbb{C}\bigl[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_d^{\pm1}, \xi_1, \xi_2, \ldots, \xi_d\bigr],$$

where ξ_j is the element represented by ∂_j. Since each $D_k(R_A)$ is $\mathbb{Z}^d$-graded—$D_k(R_A) = \bigoplus_{\mathbf{d}\in\mathbb{Z}^d} D_k(R_A) \cap D(R_A)_{\mathbf{d}}$—the graded ring $\mathrm{Gr}(D(R_A))$ inherits the grading:

$$\mathrm{Gr}\bigl(D(R_A)\bigr) = \bigoplus_{\mathbf{d}\in\mathbb{Z}^d} \mathrm{Gr}\bigl(D(R_A)\bigr)_{\mathbf{d}}.$$

In [12], we conjectured that $\mathrm{Gr}(D(R_A))$ is finitely generated $\Leftrightarrow$ R_A is a scored semigroup algebra. We proved the $\Rightarrow$ direction [12, Theorem 3.2.12]. We also proved the $\Leftarrow$ direction when the cone $\mathbb{R}_{\geqslant 0}A$ is generated by linearly independent vectors [12, Theorem 3.2.13].

2.3. Scored semigroups

Finally, we recall the definition of scored semigroups. To this end, let us define the primitive integral support function of a facet (maximal face) of the cone $\mathbb{R}_{\geqslant 0}A$. We denote by $\mathcal{F}$ the set of facets of the cone $\mathbb{R}_{\geqslant 0}A$. Given $\sigma \in \mathcal{F}$, we denote by F_σ the *primitive integral support function* of σ, i.e., F_σ is a uniquely determined linear form on $\mathbb{R}^d$ satisfying

(1) $F_\sigma(\mathbb{R}_{\geqslant 0}A) \geqslant 0$,
(2) $F_\sigma(\sigma) = 0$,
(3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Example 2.2. Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then

$$\mathcal{F} = \left\{ \begin{matrix} \sigma_{23} = \mathbb{R}_{\geqslant 0}\mathbf{a}_2 + \mathbb{R}_{\geqslant 0}\mathbf{a}_3, & \sigma_{24} = \mathbb{R}_{\geqslant 0}\mathbf{a}_2 + \mathbb{R}_{\geqslant 0}\mathbf{a}_4, \\ \sigma_{13} = \mathbb{R}_{\geqslant 0}\mathbf{a}_1 + \mathbb{R}_{\geqslant 0}\mathbf{a}_3, & \sigma_{14} = \mathbb{R}_{\geqslant 0}\mathbf{a}_1 + \mathbb{R}_{\geqslant 0}\mathbf{a}_4 \end{matrix} \right\}$$

and

$$F_{\sigma_{23}}(\theta) = \theta_1, \quad F_{\sigma_{24}}(\theta) = \theta_1 + \theta_3, \quad F_{\sigma_{13}}(\theta) = \theta_2, \quad F_{\sigma_{14}}(\theta) = \theta_2 + \theta_3,$$

where we denote the standard coordinate functions of $\mathbb{R}^d = \mathbb{R}^3$ by $\theta_1, \theta_2, \theta_3$ and $F_{\sigma_{23}}(\theta)$ is shorthand for $F_{\sigma_{23}}(\theta_1, \theta_2, \theta_3)$.

Definition 2.3. The semigroup $\mathbb{N}A$ is said to be *scored* if

$$\mathbb{N}A = \bigcap_{\sigma \in \mathcal{F}} \bigl\{\mathbf{a} \in \mathbb{Z}^d\colon\ F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A)\bigr\}. \tag{2}$$

Example 2.4. Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Then

$$\mathcal{F} = \{\sigma_1 = \mathbb{R}_{\geqslant 0}\mathbf{a}_1,\ \sigma_3 = \mathbb{R}_{\geqslant 0}\mathbf{a}_3\},$$

$F_{\sigma_1}(\theta_1, \theta_2) = \theta_2$, $F_{\sigma_3}(\theta_1, \theta_2) = 3\theta_1 - \theta_2$, and

$$\mathbb{N} \setminus F_{\sigma_1}(\mathbb{N}A) = \{1\}, \qquad \mathbb{N} \setminus F_{\sigma_3}(\mathbb{N}A) = \emptyset.$$

As illustrated in Fig. 1, the semigroup $\mathbb{N}A$ is scored.

Remark 2.5.

(1) By the definition of F_σ, the difference $\mathbb{N} \setminus F_\sigma(\mathbb{N}A)$ is finite for any $\sigma \in \mathcal{F}$.
(2) Let $\mathrm{RH}(A) = \mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d$ denote the real hull of A and let $\mathrm{Holes}(A) = \mathrm{RH}(A) \setminus \mathbb{N}A$. Then the semigroup $\mathbb{N}A$ is scored if and only if

$$\mathrm{Holes}(A) = \bigcup_{\sigma \in \mathcal{F}} \bigcup_{m \in \mathbb{N} \setminus F_\sigma(\mathbb{N}A)} F_\sigma^{-1}(m) \cap \mathrm{RH}(A).$$

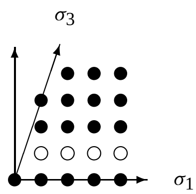

Fig. 1. The semigroup $\mathbb{N}A$ in Example 2.4 is scored.

(3) For the semigroup ring $\mathbb{C}[\mathbb{N}A]$, neither the scored property nor the Cohen–Macaulay property implies the other as shown in Examples 2.7 and 2.8, although scored semigroup rings satisfy Serre's condition (S_2) as shown in Proposition 2.6.

The semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay if and only if it satisfies Serre's condition (S_2) and the reduced homology modules of certain simplicial complexes vanish [16, Theorem 4.1]. In our case, Serre's (S_2) condition can be stated as

$$\mathbb{N}A = \bigcap_{\sigma\in\mathcal{F}} \bigl(\mathbb{N}A + \mathbb{Z}(A\cap\sigma)\bigr). \tag{S_2}$$

Proposition 2.6. *Any scored semigroup satisfies* (S_2).

Proof. Let $\mathbb{N}A$ be a scored semigroup. It is enough to show that for any facet $\sigma\in\mathcal{F}$ we have

$$\mathbb{N}A + \mathbb{Z}(A\cap\sigma) = \bigl\{\mathbf{a}\in\mathbb{Z}^d\colon\ F_\sigma(\mathbf{a})\in F_\sigma(\mathbb{N}A)\bigr\}. \tag{3}$$

The inclusion '$\subset$' is clear from the definition of F_σ. To prove the other inclusion '$\supset$', let $\mathbf{a}\in\mathbb{Z}^d$ satisfy $F_\sigma(\mathbf{a})\in F_\sigma(\mathbb{N}A)$. For every $\sigma'\in\mathcal{F}$ different from σ, there exists $\mathbf{a}_i\in A$ such that $\mathbf{a}_i\notin\sigma'$ and $\mathbf{a}_i\in\sigma$. Since $F_{\sigma'}(\mathbf{a}_i)>0$ and $\mathbb{N}\setminus F_{\sigma'}(\mathbb{N}A)$ is finite, there exists $m_i\in\mathbb{N}$ such that $F_{\sigma'}(\mathbf{a}+m_i\mathbf{a}_i)\in F_{\sigma'}(\mathbb{N}A)$.

Doing this argument for every $\sigma'\in\mathcal{F}$ different from σ, we find $\mathbf{b}\in\mathbb{N}(A\cap\sigma)$ such that

$$F_{\sigma'}(\mathbf{a}+\mathbf{b})\in F_{\sigma'}(\mathbb{N}A)\quad \bigl(\forall\sigma'\in\mathcal{F}\setminus\{\sigma\}\bigr).$$

Since

$$F_\sigma(\mathbf{a}+\mathbf{b}) = F_\sigma(\mathbf{a})\in F_\sigma(\mathbb{N}A)$$

and $\mathbb{N}A$ is scored, we see $\mathbf{a}+\mathbf{b}\in\mathbb{N}A$. Hence $\mathbf{a}\in\mathbb{N}A+\mathbb{Z}(A\cap\sigma)$. $\square$

Example 2.7. Let

$$A = \begin{pmatrix} 0 & 1 & 2\\ 1 & 1 & 0\end{pmatrix}.$$

This A satisfies (S_2). Hence the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is also Cohen–Macaulay since $\mathrm{RH}(A)$ is simplicial. Thus $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay but $\mathbb{N}A$ is not scored (see Fig. 2).

Example 2.8. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 2 & 3 & 0 & 2 & 3\\ 0 & 0 & 0 & 1 & 1 & 1\end{pmatrix}.$$

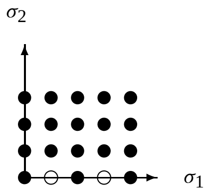

Fig. 2. The semigroup $\mathbb{N}A$ in Example 2.7 is not scored.

Then the semigroup $\mathbb{N}A$ is clearly scored. However, the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is not Cohen–Macaulay [16, Example 3.9].

3. Graded associated primes

In this section we describe the holes of the semigroup $\mathbb{N}A$, $\mathrm{Holes}(A) = \mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d \setminus \mathbb{N}A$, using the graded associated primes of certain $\mathbb{Z}^d$-graded modules.

A module M over $R := R_A = \mathbb{C}[\mathbb{N}A]$ is said to be $\mathbb{Z}^d$-*graded* if M has a decomposition $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^d} M_{\mathbf{a}}$ such that $R_{\mathbf{a}}M_{\mathbf{b}} \subset M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}$ and $\mathbf{b}$.

First, we recall a lemma from [5].

Lemma 3.1 [5, Proposition 1.3]. *The set of $\mathbb{Z}^d$-graded prime ideals of $R = \mathbb{C}[\mathbb{N}A]$ equals*

$$\bigl\{P_\tau := \mathbb{C}\bigl[\mathbb{N}A \setminus \mathbb{N}(A\cap\tau)\bigr]\colon\ \tau \text{ is a face of } \mathbb{R}_{\geqslant 0}A\bigr\}.$$

We also have the following lemma.

Lemma 3.2 (see, e.g., [1, Exercise 3.5]). *Let M be a $\mathbb{Z}^d$-graded R-module. Then any associated prime of M is $\mathbb{Z}^d$-graded, and is the annihilator of a homogeneous element.*

Lemma 3.3. *The R-module $\mathbb{C}[\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d]$ is finitely generated.*

Proof. Choose a finite subset $G \subseteq \mathbb{N}A$ that generates the cone $\mathbb{R}_{\geqslant 0}A$. Then

$$\left\{\sum_{\mathbf{a}\in G} c_{\mathbf{a}}\mathbf{a} \in \mathbb{Z}^d\colon\ 0 \leqslant c_{\mathbf{a}} < 1\right\}$$

generates $\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d$ as an $\mathbb{N}A$-set. □

Proposition 3.4. *There exist $m \in \mathbb{N}$, $\mathbf{b}_i \in \mathbb{Z}^d$, and faces τ_i of $\mathbb{R}_{\geqslant 0}A$ with $i = 1, 2, \ldots, m$ such that*

$$\mathrm{Holes}(A) = \coprod_{i=1}^{m} \bigl(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)\bigr). \tag{4}$$

Proof. Put $N := \mathbb{C}[\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d]$ and $M_0 := \mathbb{C}[\mathbb{N}A]$. Suppose $N/M_0 \neq 0$. Then $\operatorname{Ass}(N/M_0) \neq \emptyset$. Hence, by Lemma 3.2, there exists $x_1 \in N_{\mathbf{b}_1}$ such that $P_1 := \operatorname{Ann}(\overline{x_1}) \in \operatorname{Ass}(N/M_0)$. By Lemma 3.1, there exists a unique face τ_1 such that $P_1 = P_{\tau_1}$; equivalently

$$R/P_{\tau_1}[-\mathbf{b}_1] \simeq R\overline{x_1} \subset N/M_0,$$

where $R/P_{\tau_1}[-\mathbf{b}_1]$ is the $\mathbb{Z}^d$-graded module shifted by $-\mathbf{b}_1$, i.e.,

$$\bigl(R/P_{\tau_1}[-\mathbf{b}_1]\bigr)_{\mathbf{a}} := (R/P_{\tau_1})_{\mathbf{a}-\mathbf{b}_1}.$$

Hence we obtain

$$\bigl(\mathbf{b}_1 + \mathbb{N}(A \cap \tau_1)\bigr) \cap \mathbb{N}A = \emptyset.$$

Put $M_1 := M_0 + Rx_1$. If $N/M_1 \neq 0$, then there exist $\mathbf{b}_2 \in \mathbb{Z}^d$, $x_2 \in N_{\mathbf{b}_2}$, and a face τ_2 such that $P_{\tau_2} = \operatorname{Ann}(\overline{x_2}) \in \operatorname{Ass}(N/M_1)$. Since $R/P_{\tau_2}[-\mathbf{b}_2] \simeq R\overline{x_2} \subset N/M_1$, we obtain

$$\bigl(\mathbf{b}_2 + \mathbb{N}(A \cap \tau_2)\bigr) \cap \Bigl(\mathbb{N}A \coprod \bigl(\mathbf{b}_1 + \mathbb{N}(A \cap \tau_1)\bigr)\Bigr) = \emptyset.$$

Put $M_2 := M_1 + Rx_2$, repeat this process, and obtain a strictly increasing sequence of graded submodules of N: $M_0 \subset M_1 \subset M_2 \subset \cdots$. This sequence must stop since N is a Noetherian R-module (by Lemma 3.3). Thus we obtain (4). □

Note that the expression (4) is not unique. *We fix an expression* (4) *once and for all*.

Put

$$M := \max\bigl\{F_\sigma(\mathbf{b}_i):\ \sigma,\ i\bigr\} + 1. \tag{5}$$

When $\mathbb{N}A$ is normal, i.e., $\mathbb{N}A = \mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d$, we put $M = 0$. Note that for all $\sigma \in \mathcal{F}$,

$$\{k \in \mathbb{Z}:\ k \geqslant M\} \subset F_\sigma(\mathbb{N}A), \tag{6}$$

or equivalently

$$\mathbb{N} \subset -M + F_\sigma(\mathbb{N}A). \tag{7}$$

Example 3.5 (Continuation of Example 2.4). Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Then

$$\mathbb{N} \setminus F_{\sigma_1}(\mathbb{N}A) = \{1\}, \qquad \mathbb{N} \setminus F_{\sigma_3}(\mathbb{N}A) = \emptyset.$$

We have $M = 2$.

Lemma 3.6. *Let τ be a face of the cone $\mathbb{R}_{\geqslant 0}A$. Then*

$$\mathbb{N}A+\mathbb{Z}(A\cap\tau)=\left[\mathbb{R}_{\geqslant 0}A+\mathbb{R}(A\cap\tau)\right]\cap\mathbb{Z}^d \Big\backslash \bigcup_{\tau_i\succcurlyeq\tau}\left(\mathbf{b}_i+\mathbb{Z}(A\cap\tau_i)\right).$$

In particular, if $\mathbf{c}\in\mathbb{Z}^d$ satisfies $F_\sigma(\mathbf{c})\geqslant M$ for all $\sigma\succcurlyeq\tau$, then $\mathbf{c}\in\mathbb{N}A+\mathbb{Z}(A\cap\tau)$.

Proof. $\subseteq$: Let $\mathbf{d}\in\mathbb{N}A+\mathbb{Z}(A\cap\tau)$. Then there exists $\mathbf{d}_\tau\in\mathbb{N}(A\cap\tau)$ such that $\mathbf{d}+\mathbf{d}_\tau\in\mathbb{N}A$. Suppose that $\mathbf{d}\in\mathbf{b}_i+\mathbb{Z}(A\cap\tau_i)$ for some $\tau_i\succcurlyeq\tau$. Then there exists $\mathbf{d}_i\in\mathbb{N}(A\cap\tau_i)$ such that $\mathbf{d}+\mathbf{d}_i\in\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)$. Since $\tau\preccurlyeq\tau_i$, we have $\mathbf{d}+\mathbf{d}_i+\mathbf{d}_\tau\in\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)$. However $(\mathbf{d}+\mathbf{d}_\tau)+\mathbf{d}_i\in\mathbb{N}A$ which leads to a contradiction because $(\mathbf{b}_i+\mathbb{Z}(A\cap\tau))\cap\mathbb{N}A=\emptyset$ by Proposition 3.4.

$\supseteq$: Assume $\mathbf{d}\in[\mathbb{R}_{\geqslant 0}A+\mathbb{R}(A\cap\tau)]\cap\mathbb{Z}^d\setminus(\mathbb{N}A+\mathbb{Z}(A\cap\tau))$. Because $\mathbf{d}\in(\mathbb{R}_{\geqslant 0}A+\mathbb{R}(A\cap\tau))\cap\mathbb{Z}^d$, there exists $\mathbf{d}_\tau\in\mathbb{N}(A\cap\tau)$ such that $\mathbf{d}+\mathbf{d}_\tau\in\mathbb{R}_{\geqslant 0}A\cap\mathbb{Z}^d$. Since $\mathbf{d}\notin\mathbb{N}A+\mathbb{Z}(A\cap\tau)$, $\mathbf{d}+\mathbf{d}_\tau\notin\mathbb{N}A$. So by Proposition 3.4, $\mathbf{d}+\mathbf{d}_\tau\in\mathbf{b}_j+\mathbb{N}(A\cap\tau_j)$ for some τ_j. We claim that we can take the above $\mathbf{d}_\tau$ so that $\mathbf{d}+\mathbf{d}_\tau\in\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)$ for some $\tau_i\succcurlyeq\tau$. To prove the claim it is enough to show that we can take $\mathbf{d}_\tau$ so that $\mathbf{d}+\mathbf{d}_\tau\notin\mathbf{b}_j+\mathbb{N}(A\cap\tau_j)$ for any $\tau_j\not\succcurlyeq\tau$. For each $\tau_j\not\succcurlyeq\tau$, there exists a facet σ_j with $\sigma_j\not\succcurlyeq\tau$ and $\sigma_j\succcurlyeq\tau_j$. Take a vector $\mathbf{d}_j\in\mathbb{N}(A\cap\tau)\setminus\mathbb{N}(A\cap\sigma_j)$. Set $\mathbf{d}'_\tau=\mathbf{d}_\tau+M\sum\mathbf{d}_j$, where the sum is over all faces τ_j not containing τ. Then $\mathbf{d}+\mathbf{d}'_\tau\notin\mathbf{b}_j+\mathbb{N}(A\cap\tau_j)$ if $\tau_j\not\succcurlyeq\tau$ because there exists a facet σ_j containing τ_j with $F_{\sigma_j}(\mathbf{d}+\mathbf{d}'_\tau)=F_{\sigma_j}(\mathbf{d}+\mathbf{d}_\tau+M\sum\mathbf{d}_k)\geqslant M$. $\square$

4. Decomposition of the lattice $\mathbb{Z}^d$

In this section, we decompose the lattice $\mathbb{Z}^d$ into finite pieces suitable for the finite-generation arguments in Sections 5 and 6.

The ring $D(R)$ localizes well: $D(S^{-1}R)=S^{-1}R\otimes_R D(R)$ [12, Lemma 3.2.1]. This allows us to reduce structural questions about $D(\mathbb{C}[\mathbb{N}A])$ to the case where the cone $\sigma=\mathbb{R}_{\geqslant 0}A$ generated by the columns of A is strongly convex (σ does not contain any lines through the origin). If the cone σ is not strongly convex then $\mathbb{N}A$ contains a finite subset B so that $\mathbb{R}_{\geqslant 0}B$ is strongly convex and $R_A=\mathbb{C}[\mathbb{N}A]$ is a localization of $\mathbb{C}[\mathbb{N}B]$. *From now on we assume that the cone $\mathbb{R}_{\geqslant 0}A$ is strongly convex.*

We call a cone $\rho=\mathbb{R}_{\geqslant 0}\mathbf{v}_\rho$ a *ray* of the hyperplane arrangement determined by A if $\mathbf{v}_\rho$ is a nonzero integral vector, and $\mathbb{R}\rho$ is an intersection of hyperplanes $(F_\sigma=0)$ $(\sigma\in\mathcal{F})$. Let $\mathrm{Ray}(A)$ denote the set of rays of the hyperplane arrangement determined by A. Let $\rho\in\mathrm{Ray}(A)$, and let $\mathbf{e}_\rho$ be the generator of $\mathbb{Z}^d\cap\rho$, i.e., $\mathbb{Z}^d\cap\rho=\mathbb{N}\mathbf{e}_\rho$.

Set

$$\mathrm{Facet}_+(\rho):=\left\{\sigma\in\mathcal{F}\colon\ F_\sigma(\mathbf{e}_\rho)>0\right\},\qquad \mathrm{Facet}_0(\rho):=\left\{\sigma\in\mathcal{F}\colon\ F_\sigma(\rho)=0\right\},$$
$$\mathrm{Facet}_-(\rho):=\left\{\sigma\in\mathcal{F}\colon\ F_\sigma(\mathbf{e}_\rho)<0\right\}.$$

Let M be the nonnegative integer defined by (5). For a ray $\rho\in\mathrm{Ray}(A)$, take a nonzero vector $\mathbf{d}_\rho$ from $\mathbb{Z}^d\cap\rho$ satisfying the condition:

$$F_\sigma(\mathbf{d}_\rho) \geqslant M \quad \text{if } \sigma \in \mathrm{Facet}_+(\rho), \qquad F_\sigma(\mathbf{d}_\rho) = 0 \quad \text{if } \sigma \in \mathrm{Facet}_0(\rho),$$
$$F_\sigma(\mathbf{d}_\rho) \leqslant -M \quad \text{if } \sigma \in \mathrm{Facet}_-(\rho). \tag{8}$$

Note that the second condition above is automatically satisfied since $\mathbf{d}_\rho \in \rho$.

Example 4.1 (Continuation of Example 2.2). Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Since $\mathbb{N}A$ is normal, $M = 0$. We have

$$\begin{aligned} &(F_{\sigma_{23}} = 0) \cap (F_{\sigma_{13}} = 0) = \mathbb{R}\,{}^t(0,0,1), && (F_{\sigma_{23}} = 0) \cap (F_{\sigma_{24}} = 0) = \mathbb{R}\,{}^t(0,1,0), \\ &(F_{\sigma_{23}} = 0) \cap (F_{\sigma_{14}} = 0) = \mathbb{R}\,{}^t(0,1,-1), && (F_{\sigma_{13}} = 0) \cap (F_{\sigma_{24}} = 0) = \mathbb{R}\,{}^t(1,0,-1), \\ &(F_{\sigma_{13}} = 0) \cap (F_{\sigma_{14}} = 0) = \mathbb{R}\,{}^t(1,0,0), && (F_{\sigma_{24}} = 0) \cap (F_{\sigma_{14}} = 0) = \mathbb{R}\,{}^t(1,1,-1). \end{aligned}$$

Hence we can take

$$\{\mathbf{d}_\rho \colon \rho \in \mathrm{Ray}(A)\} = \left\{ \begin{array}{c} \pm^t(0,0,1),\ \pm^t(0,1,0),\ \pm^t(0,1,-1) \\ \pm^t(1,0,-1),\ \pm^t(1,0,0),\ \pm^t(1,1,-1) \end{array} \right\}.$$

Note that $\mathrm{Ray}(A)$ has more elements than

$$\{\pm(\text{1-dimensional face of } \mathbb{R}_{\geqslant 0}A)\}.$$

We now decompose the lattice $\mathbb{Z}^d$ into pieces. Let μ be a map from $\mathcal{F}$ to a set

$$\widetilde{M} := \{-\infty\} \cup \{+\infty\} \cup \{m \in \mathbb{Z} \colon |m| < M\}.$$

Define a subset S_μ of $\mathbb{Z}^d$ by

$$S_\mu := \{\mathbf{d} \in \mathbb{Z}^d \colon F_\sigma(\mathbf{d}) = \mu(\sigma) \text{ for all } \sigma \in \mathcal{F}\}, \tag{9}$$

where we agree that $F_\sigma(\mathbf{d}) = +\infty$ ($-\infty$, respectively) means $F_\sigma(\mathbf{d}) \geqslant M$ ($\leqslant -M$, respectively). Note that S_μ could be empty. Clearly, we have

$$\mathbb{Z}^d = \bigcup_\mu S_\mu.$$

The S_μ are the integral points (represented by large dots) in the shaded regions in Fig. 3; see Example 4.3.

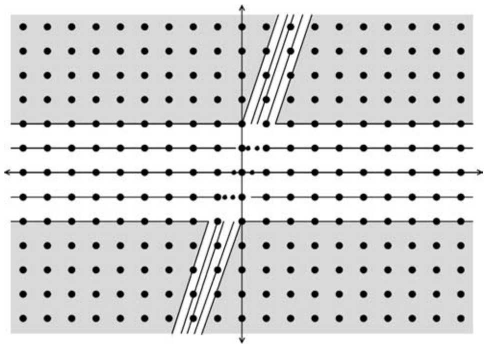

Fig. 3. The sets $S_{\mu,\mathbb{R}}$.

Example 4.2 (Continuation of Example 3.5). In the example, $M = 2$ and $\widetilde{M} = \{\pm\infty\} \cup \{-1, 0, 1\}$. We can take

$$\{\mathbf{d}_\rho \colon \rho \in \mathrm{Ray}(A)\} = \{\pm\mathbf{a}_1, \pm\mathbf{a}_3\}.$$

Consider the following maps μ_1 and μ_2:

$$\mu_1(\sigma_1) = 1\mu_1(\sigma_3) = -1, \qquad \mu_2(\sigma_1) = 1\mu_2(\sigma_3) = -\infty.$$

Then

$$S_{\mu_1} = \{\mathbf{d} \in \mathbb{Z}^2 \colon d_2 = 1,\ 3d_1 - d_2 = -1\} = \{{}^t(0, 1)\},$$
$$S_{\mu_2} = \{\mathbf{d} \in \mathbb{Z}^2 \colon d_2 = 1,\ 3d_1 - d_2 \leqslant -2\} = \mathbb{N}(-\mathbf{a}_1) + {}^t(-1, 1).$$

We also construct similar subsets of $\mathbb{R}^d$; set

$$S_{\mu,\mathbb{R}} := \{\mathbf{d} \in \mathbb{R}^d \colon F_\sigma(\mathbf{d}) = \mu(\sigma) \text{ for all } \sigma \in \mathcal{F}\},$$
$$F_{\mu,\mathbb{R}} := \bigcap_\sigma \left\{ \mathbf{d} \in \mathbb{R}^d \colon \begin{array}{ll} F_\sigma(\mathbf{d}) = 0 & \text{if } \mu(\sigma) \neq \pm\infty \\ F_\sigma(\mathbf{d}) \geqslant 0 & \text{if } \mu(\sigma) = +\infty \\ F_\sigma(\mathbf{d}) \leqslant 0 & \text{if } \mu(\sigma) = -\infty \end{array} \right\}, \tag{10}$$

and

$$F_\mu := \{\mathbf{d}_\rho \colon \rho \subset F_{\mu,\mathbb{R}}\}.$$

For example, if μ is the constant function $+\infty$, then $F_{\mu,\mathbb{R}} = \mathbb{R}_{\geqslant 0}A$. We have

$$S_\mu = \mathbb{Z}^d \cap S_{\mu,\mathbb{R}}.$$

Example 4.3 (Continuation of Example 4.2).

In our example,

$$F_{\mu_1} = \emptyset, \qquad F_{\mu_2,\mathbb{R}} = \{\mathbf{d} \in \mathbb{R}^2\colon\ d_2 = 0,\ 3d_1 - d_2 \leqslant 0\} = \{\mathbf{d} \in \mathbb{R}^2\colon\ d_2 = 0,\ d_1 \leqslant 0\},$$
$$F_{\mu_2} = \{-\mathbf{a}_1\}, \qquad S_{\mu_1,\mathbb{R}} = \{{}^t(0,1)\}, \qquad S_{\mu_2,\mathbb{R}} = F_{\mu_2,\mathbb{R}} + {}^t(-1/3, 1).$$

Lemma 4.4. *Let V_μ denote the set of vertices of the polyhedron $S_{\mu,\mathbb{R}}$. Then*

$$S_{\mu,\mathbb{R}} = F_{\mu,\mathbb{R}} + \mathrm{conv}(V_\mu),$$

where $\mathrm{conv}(V_\mu)$ *denotes the convex hull of V_μ.*

Proof. $F_{\mu,\mathbb{R}}$ is the characteristic cone of $S_{\mu,\mathbb{R}}$. See [13, §8.9 (28)]. □

Lemma 4.5. $F_{\mu,\mathbb{R}} = \mathbb{R}_{\geqslant 0} F_\mu$.

Proof. This follows from the fact that a strongly convex cone is generated by its 1-dimensional faces. □

Proposition 4.6. *The set S_μ is F_μ-finite, i.e., there exist $\mathbf{v}_1, \ldots, \mathbf{v}_r \in S_\mu$ such that $S_\mu = \bigcup_{j=1}^r ((\mathbb{N}F_\mu) + \mathbf{v}_j)$, where $\mathbb{N}F_\mu = \sum_{\mathbf{u}\in F_\mu} \mathbb{N}\mathbf{u}$.*

Proof. Let

$$G_\mu := \left(\left\{\sum_{\mathbf{u}\in F_\mu} a_{\mathbf{u}}\mathbf{u}\colon\ 0 \leqslant a_{\mathbf{u}} < 1\right\} + \mathrm{conv}(V_\mu)\right) \cap \mathbb{Z}^d.$$

Then G_μ is a finite set. Let $G_\mu = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Clearly $S_\mu \supseteq \bigcup_{j=1}^r ((\mathbb{N}F_\mu) + \mathbf{v}_j)$.

Now suppose that $\mathbf{v} \in S_\mu$. Then, by Lemmas 4.4 and 4.5, there exist $c_{\mathbf{u}} \in \mathbb{R}_{\geqslant 0}$ and $\mathbf{w} \in \mathrm{conv}(V_\mu)$ such that $\mathbf{v} = \sum_{\mathbf{u}\in F_\mu} c_{\mathbf{u}}\mathbf{u} + \mathbf{w}$. Hence

$$\mathbf{v} = \sum_{\mathbf{u}\in F_\mu} \lfloor c_{\mathbf{u}} \rfloor \mathbf{u} + \left(\sum_{\mathbf{u}\in F_\mu} \bigl(c_{\mathbf{u}} - \lfloor c_{\mathbf{u}} \rfloor\bigr)\mathbf{u} + \mathbf{w}\right) \in \bigcup_{j=1}^r \bigl((\mathbb{N}F_\mu) + \mathbf{v}_j\bigr). \qquad \square$$

Example 4.7 (Continuation of Example 4.3). In the example,

$$G_{\mu_1} = \{{}^t(0,1)\} \quad \text{and} \quad G_{\mu_2} = \{-c\mathbf{a}_1 + {}^t(-1/3, 1) \in \mathbb{Z}^2\colon\ 0 \leqslant c < 1\} = \{{}^t(-1,1)\}.$$

5. Finite generation of $D(R_A)$

In this section, we prove that $D(R_A)$ is always finitely generated.

Let $\mathbf{d} \in \mathbb{Z}^d$. Recall that $\Omega(\mathbf{d}) = \{\mathbf{a} \in \mathbb{N}A\colon\ \mathbf{a} + \mathbf{d} \notin \mathbb{N}A\}$. First, we describe the Zariski closure of the set $\Omega(\mathbf{d})$ using (4). We denote the Zariski closure of a set V in $\mathbb{C}^d$ by $\mathrm{ZC}(V)$.

Proposition 5.1.

$$\begin{aligned}\mathrm{ZC}\big(\Omega(\mathbf{d})\big) &= \big[-\mathbf{d} + \mathrm{ZC}\big((\mathbf{d}+\mathbb{N}A)\setminus \mathbb{R}_{\geqslant 0}A\big)\big] \cup \big[-\mathbf{d} + \mathrm{ZC}\big((\mathbf{d}+\mathbb{N}A)\cap \mathrm{Holes}(A)\big)\big] \\ &= \Bigg(\bigcup_{\sigma: F_\sigma(\mathbf{d})<0}\ \bigcup_{m<-F_\sigma(\mathbf{d}),\, m\in F_\sigma(\mathbb{N}A)} F_\sigma^{-1}(m)\Bigg) \\ &\quad \cup \Bigg(\bigcup_{\mathbf{b}_i-\mathbf{d}\in\mathbb{N}A+\mathbb{Z}(A\cap\tau_i)} \big(\mathbf{b}_i - \mathbf{d} + \mathbb{C}(A\cap\tau_i)\big)\Bigg),\end{aligned}$$

where $\mathrm{Holes}(A) = \coprod_{i=1}^m (\mathbf{b}_i + \mathbb{N}(A\cap\tau_i))$ *as in Proposition* 3.4.

Proof. Recall that

$$\Omega(\mathbf{d}) = -\mathbf{d} + \big[(\mathbf{d}+\mathbb{N}A)\setminus \mathbb{N}A\big]. \tag{11}$$

So we consider the set $(\mathbf{d}+\mathbb{N}A)\setminus\mathbb{N}A$. First, we have

$$\begin{aligned}(\mathbf{d}+\mathbb{N}A)\setminus\mathbb{N}A &= \big[(\mathbf{d}+\mathbb{N}A)\setminus\mathbb{R}_{\geqslant 0}A\big] && (12)\\ &\quad \cup \big[(\mathbf{d}+\mathbb{N}A)\cap \mathrm{Holes}(A)\big]. && (13)\end{aligned}$$

The Zariski closure of the first set (12) is easy to describe:

$$\mathrm{ZC}\big((\mathbf{d}+\mathbb{N}A)\setminus\mathbb{R}_{\geqslant 0}A\big) = \bigcup_{\sigma: F_\sigma(\mathbf{d})<0}\ \bigcup_{m<0,\, m\in F_\sigma(\mathbb{N}A)+F_\sigma(\mathbf{d})} F_\sigma^{-1}(m). \tag{14}$$

The second set (13) is written as

$$(\mathbf{d}+\mathbb{N}A)\cap\big(\mathrm{Holes}(A)\big) = \bigcup_i (\mathbf{d}+\mathbb{N}A)\cap\big(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)\big).$$

Note that shifting the set $(\mathbf{d}+\mathbb{N}A)\cap(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i))$ by adding elements of $\mathbb{N}(A\cap\tau_i)$ produces no new elements. Hence, if the set is not empty then its Zariski closure is

$$\mathrm{ZC}\big((\mathbf{d}+\mathbb{N}A)\cap\big(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)\big)\big) = \mathbf{b}_i + \mathbb{C}(A\cap\tau_i). \tag{15}$$

Finally, note that

$$(\mathbf{d}+\mathbb{N}A)\cap\big(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)\big)\neq\emptyset \quad\Longleftrightarrow\quad \mathbf{b}_i-\mathbf{d}\in\mathbb{N}A+\mathbb{Z}(A\cap\tau_i). \tag{16}$$

We have thus proved the proposition. □

Example 5.2 (Continuation of Example 2.7). Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then $F_{\sigma_1}(\theta) = \theta_2$, $F_{\sigma_2}(\theta) = \theta_1$, and $F_{\sigma_i}(\mathbb{N}A) = \mathbb{N}$ $(i = 1, 2)$. We have $\mathbb{N}A = \mathbb{N}^2 \setminus [{}^t(1,0) + \mathbb{N}{}^t(2,0)]$ and $M = F_{\sigma_2}({}^t(1,0)) + 1 = 2$. The semigroup $\mathbb{N}A$ is not scored.

As described in Lemma 3.6,

$$\begin{aligned} \mathbb{N}A + \mathbb{Z}(A \cap \sigma_1) &= \left\{{}^t(a_1, a_2) \in \mathbb{Z}^2\colon\ a_2 \geqslant 0\right\} \Big\backslash \left[{}^t(1,0) + \mathbb{Z}{}^t(2,0)\right] \\ &= \left\{{}^t(a_1, a_2) \in \mathbb{Z}^2\colon\ a_2 \geqslant 1\right\} \cup \left\{{}^t(a_1, 0) \in \mathbb{Z}^2\colon\ a_1 \in 2\mathbb{Z}\right\}. \end{aligned}$$

Hence ${}^t(1,0) - {}^t(d_1, d_2) \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma_1)$ if and only if $d_2 \leqslant -1$ or [$d_2 = 0$ and d_1 is odd]. Therefore, by Proposition 5.1,

$$\mathrm{ZC}\big(\Omega(\mathbf{d})\big) = \begin{cases} \mathbb{V}\left(\displaystyle\prod_{d_i<0} \prod_{m=0}^{-d_i-1} (\theta_i - m)\right) & \text{if } d_2 > 1 \text{ or } [d_2 = 0 \text{ and } d_1 \in 2\mathbb{Z}], \\ \mathbb{V}\left((\theta_2 + d_2) \cdot \displaystyle\prod_{d_i<0} \prod_{m=0}^{-d_i-1} (\theta_i - m)\right) & \text{otherwise,} \end{cases}$$

where $\mathbb{V}(f)$ is the largest subset of $\mathbb{C}^d$ on which f vanishes.

Example 5.3. Let

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \end{pmatrix}.$$

We use this example to outline our approach to showing that $D(R_A)$ is finitely generated (Fig. 4). Consider the index $\mathbf{d} = {}^t(-8, -4)$. We aim to express $D(R_A)_{\mathbf{d}}$ as $D(R_A)_{\mathbf{d}_1} D(R_A)_{\mathbf{d}_2}$ for $\mathbf{d}_1$ and $\mathbf{d}_2$ vectors of smaller norm such that $\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{d}$. For instance, if $\mathbf{d}_1 = {}^t(-6, -4)$ and $\mathbf{d}_2 = {}^t(-2, 0)$ then $D(R_A)_{\mathbf{d}_1} D(R_A)_{\mathbf{d}_2}$ is a module over the ring $D(R_A)_{\mathbf{0}} = \mathbb{C}[\theta_1, \theta_2]$. This module equals

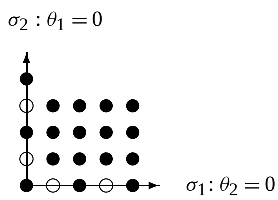

Fig. 4. The semigroup $\mathbb{N}A$ in Example 5.3.

$$\begin{aligned}
&t_1^{-6}t_2^{-4}\prod_{i=0}^{6}(\theta_1-i)\prod_{j=0}^{4}(\theta_2-j)\cdot t_1^{-2}\prod_{i=0}^{2}(\theta_1-i)\mathbb{C}[\theta_1,\theta_2]\\
&\quad=t^{\mathbf{d}}\prod_{i=0}^{8}(\theta_1-i)\prod_{j=0}^{4}(\theta_2-j)\cdot(\theta_1-2)\mathbb{C}[\theta_1,\theta_2]\\
&\quad=D(R_A)_{\mathbf{d}}(\theta_1-2).
\end{aligned}$$

So it is not possible to use just this pair $\mathbf{d}_1$ and $\mathbf{d}_2$ to generate the module $D(R_A)_{\mathbf{d}}$. We say that the pair $\mathbf{d}_1,\mathbf{d}_2$ is deficient by the ideal $\langle\theta_1-2\rangle$ of $\mathbb{C}[\theta_1,\theta_2]$. However, for the pair $\mathbf{d}'_1={}^t(-4,-4)$ and $\mathbf{d}'_2={}^t(-4,0)$, $D(R_A)_{\mathbf{d}'_1}D(R_A)_{\mathbf{d}'_2}=D(R_A)_{\mathbf{d}}(\theta_1-4)$ so

$$D(R_A)_{\mathbf{d}_1}D(R_A)_{\mathbf{d}_2}+D(R_A)_{\mathbf{d}'_1}D(R_A)_{\mathbf{d}'_2}=D(R_A)_{\mathbf{d}}\big[\langle\theta_1-2\rangle+\langle\theta_1-4\rangle\big]=D(R_A)_{\mathbf{d}}.$$

So it is possible to express $D(R_A)_{\mathbf{d}}$ as a *sum* of terms of the form $D(R_A)_{\mathbf{d}_1}D(R_A)_{\mathbf{d}_2}$. However, as in the example, we need to choose the terms $\mathbf{d}_1$ and $\mathbf{d}_2$ carefully in order that the deficiency ideals sum to the unit ideal of $D(R_A)_{\mathbf{0}}=\mathbb{C}[\theta]$.

Definition 5.4. Given $\mathbf{d}_1,\mathbf{d}_2\in\mathbb{Z}^d$ with $\mathbf{d}_1+\mathbf{d}_2=\mathbf{d}$, the *deficiency ideal* of $\mathbf{d}_1$ and $\mathbf{d}_2$ is the ideal I of $\mathbb{C}[\theta_1,\dots,\theta_d]$ such that $D(R_A)_{\mathbf{d}_1}D(R_A)_{\mathbf{d}_2}=D(R_A)_{\mathbf{d}}I$.

The example suggests that the deficiency ideal is the ideal that vanishes on the Zariski closure of a translate of some of the holes of $\mathbb{N}A$. However, the exact portion of the set of holes that is involved can depend on parity considerations. For instance, in the example, if $\mathbf{d}={}^t(-4,0)$ and $\mathbf{d}_1={}^t(-1,0)$ and $\mathbf{d}_2={}^t(-3,0)$ then the deficiency ideal is $\langle\theta_1-3\rangle\langle\theta_2\rangle^2$ while if $\mathbf{d}_1=\mathbf{d}_2={}^t(-2,0)$, it is $\langle\theta_1-2\rangle$. We handle parity concerns by choosing $\mathbf{d}_2$ carefully (as a multiple of a particularly good vector $\mathbf{d}_\rho$; see definition (8) and (17) below).

We show that the pairs $\mathbf{d}_1$ and $\mathbf{d}_2$ can be chosen so that the sum of the deficiency ideals is the unit ideal in Theorem 5.14. Moreover, we locate a good set of generators for $D(R_A)$. This requires a description of the two ideals $\mathbb{I}(\Omega(\mathbf{d}_1)+\mathbf{d}_2)$ and $\mathbb{I}(\Omega(\mathbf{d}_2))$ appearing in the expression

$$D(R_A)_{\mathbf{d}_1}D(R_A)_{\mathbf{d}_2}=t^{\mathbf{d}}\mathbb{I}\big(\Omega(\mathbf{d}_1)+\mathbf{d}_2\big)\mathbb{I}\big(\Omega(\mathbf{d}_2)\big).$$

In turn this requires some computations based on the geometry of the semigroup $\mathbb{N}A$ (Lemmas 5.6–5.12). Since S and its Zariski-closure $\mathrm{ZC}(S)$ have the same idealization, $\mathbb{I}(S)=\mathbb{I}(\mathrm{ZC}(S))$, the Zariski-closure of the sets $\Omega(\mathbf{d}_1)+\mathbf{d}_2$ and $\Omega(\mathbf{d}_2)$ play a significant role in the description of the deficiency ideal of $\mathbf{d}_1$ and $\mathbf{d}_2$. In Corollary 5.13, we describe the deficiency ideal in a form suitable for our purpose.

Let $\rho\in\mathrm{Ray}(A)$. Take $\mathbf{d}_\rho$ so that

$$\mathbf{d}_\rho\in\mathbb{Z}(A\cap\tau)\cap\rho\quad\text{for all faces }\tau\text{ of }\mathbb{R}_{\geqslant0}A\text{ satisfying }\mathbb{R}\tau\supset\rho.\tag{17}$$

Indeed, this is possible; for example, for a face τ with $\mathbb{R}\tau\supset\rho$, let m_τ be the index $[\mathbb{Z}^d\cap\mathbb{Q}(A\cap\tau):\mathbb{Z}(A\cap\tau)]$. Let m_ρ be a common multiple, not less than M, of the m_τ.

Then $\mathbf{d}_\rho := m_\rho \mathbf{e}_\rho$ satisfies the condition since $(\mathbb{Z}^d \cap \rho)/(\mathbb{Z}(A \cap \tau) \cap \rho)$ is a subgroup of $(\mathbb{Z}^d \cap \mathbb{Q}(A \cap \tau))/(\mathbb{Z}(A \cap \tau))$.

We want to show that $D(R)_{\mathbf{d}}$ equals a sum of $D(R)_{\mathbf{d}_1} \cdot D(R)_{\mathbf{d}_2}$ for some choices of $\mathbf{d}_2$, where $\mathbf{d}_1 = \mathbf{d} - \mathbf{d}_2$. In Theorem 5.14 we will choose the $\mathbf{d}_2$'s to be multiples of $\mathbf{d}_\rho$. Until then, we concentrate on what happens when $\mathbf{d}_2 = \mathbf{d}_\rho$.

Note that $D(R)_{\mathbf{d}_1} \cdot D(R)_{\mathbf{d}_\rho} = t^{\mathbf{d}} \mathbb{I}(X)\mathbb{I}(Y)$, where

$$X := (-\mathbf{d}_\rho + \mathbb{N}A) \setminus (-\mathbf{d} + \mathbb{N}A) = -\mathbf{d} + \big[(\mathbf{d}_1 + \mathbb{N}A) \setminus \mathbb{N}A\big], \tag{18}$$

$$Y := \mathbb{N}A \setminus (-\mathbf{d}_\rho + \mathbb{N}A) = \Omega(\mathbf{d}_\rho). \tag{19}$$

Remark 5.5. If $\rho \subset F_{\mu,\mathbb{R}}$ (see Section 4 for the notation), then

$$\mathrm{Facet}_+(\rho) \subset \mu^{-1}(+\infty), \qquad \mathrm{Facet}_-(\rho) \subset \mu^{-1}(-\infty).$$

Lemma 5.6. *Let* $\mathbf{d}_1 \in S_\mu$, $\rho \subset F_{\mu,\mathbb{R}}$, *and* $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_\rho$. *Then*

$$\mathrm{ZC}\big((\mathbf{d} + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big)\big) = \mathrm{ZC}\big((\mathbf{d}_1 + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big)\big) \tag{20}$$

for all i.

Proof. By (15) it is enough to show that

$$(\mathbf{d} + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) = \emptyset \quad \Longleftrightarrow \quad (\mathbf{d}_1 + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) = \emptyset.$$

This is equivalent to saying that

$$\mathbf{b}_i - \mathbf{d} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i) \quad \Longleftrightarrow \quad \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$$

by (16). We divide the proof into several cases.

Case I. There exists $\sigma \in \mathrm{Facet}_+(\rho)$ such that $\sigma \not\succeq \tau_i$.

Then, by Remark 5.5, $F_\sigma(\mathbf{d}), F_\sigma(\mathbf{d}_1) \geqslant M$. We claim that both $(\mathbf{d} + \mathbb{N}A) \cap (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i))$ and $(\mathbf{d}_1 + \mathbb{N}A) \cap (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i))$ are empty. Suppose that the set $(\mathbf{d} + \mathbb{N}A) \cap (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i))$ is not empty. Let $\mathbf{x} \in (\mathbf{d} + \mathbb{N}A) \cap (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i))$. Since $\mathbf{x}$ belongs to $(\mathbf{d} + \mathbb{N}A)$, we have $F_\sigma(\mathbf{x}) \geqslant F_\sigma(\mathbf{d}) \geqslant M$. Since $\sigma \not\succeq \tau_i$ and $\mathbf{x} \in \mathbf{b}_i + \mathbb{N}(A \cap \tau_i)$, we have $F_\sigma(\mathbf{x}) = F_\sigma(\mathbf{b}_i)$, and this is less than M by the definition of M (5). We thus have a contradiction. Hence, the set $(\mathbf{d} + \mathbb{N}A) \cap (\mathbf{b}_i + \mathbb{N}(A \cap \tau_i))$ is empty. The same argument works for $\mathbf{d}_1$, too.

Case II. $\sigma \in \mathrm{Facet}_-(\rho) \cup \mathrm{Facet}_0(\rho)$ for all $\sigma \not\succeq \tau_i$.

If a face τ satisfies $\mathbb{R}\tau \supset \rho$, then $\mathbf{d}_\rho \in \mathbb{Z}(A \cap \tau)$ by (17). Hence

$$\mathbf{b}_i - \mathbf{d} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau) \quad \Longleftrightarrow \quad \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau). \tag{21}$$

We divide Case II into three subcases: II-a, II-a′, and II-b.

Case II-a. There exists a face $\tau \succcurlyeq \tau_i$ such that $\mathbb{R}\tau \supset \rho$ and $\mathbf{b}_i - \mathbf{d} \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau)$.

In this case, we have $\mathbf{b}_i - \mathbf{d}_1 \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau)$ by (21). Hence $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$. Hence, we obtain

$$(\mathbf{d} + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) = \emptyset = (\mathbf{d}_1 + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big).$$

Case II-a′. There exists a face $\tau \succcurlyeq \tau_i$ such that $\mathbb{R}\tau \supset \rho$ and $\mathbf{b}_i - \mathbf{d}_1 \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau)$.

Similarly to Case II-a, we obtain

$$(\mathbf{d} + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) = \emptyset = (\mathbf{d}_1 + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big).$$

Case II-b. $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)$ for all faces τ satisfying $\tau \succcurlyeq \tau_i$ and $\mathbb{R}\tau \supset \rho$.

In this case, we prove that

$$(\mathbf{d} + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) \neq \emptyset, \qquad (\mathbf{d}_1 + \mathbb{N}A) \cap \big(\mathbf{b}_i + \mathbb{N}(A \cap \tau_i)\big) \neq \emptyset,$$

or equivalently

$$\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i). \tag{22}$$

To prove (22), we use Lemma 3.6; we first claim that

$$\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{R}_{\geqslant 0}A + \mathbb{R}(A \cap \tau_i), \tag{23}$$

and we next claim that

$$\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \notin \mathbf{b}_j + \mathbb{Z}(A \cap \tau_j) \quad (\forall \tau_j \succcurlyeq \tau_i). \tag{24}$$

To prove (23), since $\mathbb{R}_{\geqslant 0}A + \mathbb{R}(A \cap \tau_i) = \bigcap_{\sigma \succcurlyeq \tau_i, \sigma \in \mathcal{F}} (F_\sigma \geqslant 0)$, it is enough to show that $F_\sigma(\mathbf{b}_i - \mathbf{d}), F_\sigma(\mathbf{b}_i - \mathbf{d}_1) \geqslant 0$ for all facets $\sigma \succcurlyeq \tau_i$.

If a facet σ satisfies $\sigma \succcurlyeq \tau_i$ and $\mathbb{R}\sigma \supset \rho$, then $F_\sigma(\mathbf{b}_i - \mathbf{d}), F_\sigma(\mathbf{b}_i - \mathbf{d}_1) \geqslant 0$, since $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \sigma)$ by our assumption II-b.

If a facet $\sigma \succcurlyeq \tau_i$ does not satisfy $\mathbb{R}\sigma \supset \rho$, then by our assumption for Case II, we have $\sigma \in \mathrm{Facet}_-(\rho)$, since $\mathbb{R}\sigma \supset \rho \Leftrightarrow \sigma \in \mathrm{Facet}_0(\rho)$. Now, by the definition of $\mathbf{d}_\rho$ (8), $F_\sigma(\mathbf{d}_\rho) \leqslant -M$. Since $\mathbf{d}_1 \in S_\mu$, Remark 5.5 implies $F_\sigma(\mathbf{d}_1) \leqslant -M$. Hence $F_\sigma(\mathbf{b}_i - \mathbf{d}_1) = F_\sigma(\mathbf{b}_i) - F_\sigma(\mathbf{d}_1) \geqslant M \geqslant 0$ and $F_\sigma(\mathbf{b}_i - \mathbf{d}) = F_\sigma(\mathbf{b}_i - \mathbf{d}_1) - F_\sigma(\mathbf{d}_\rho) \geqslant 2M \geqslant 0$. We have thus proved the claim (23).

Next we prove the claim (24). Suppose that $\tau_j \succcurlyeq \tau_i$ satisfies $\mathbb{R}\tau_j \supset \rho$. Then, by our assumption II-b, $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_j)$. Hence, by Lemma 3.6, we have $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \notin \mathbf{b}_j + \mathbb{Z}(A \cap \tau_j)$.

Next suppose that $\tau_j \succcurlyeq \tau_i$ does not satisfy $\mathbb{R}\tau_j \supset \rho$. Then there exists a facet $\sigma \succcurlyeq \tau_j$ such that $\mathbb{R}\sigma \not\supset \rho$ by the Sublemma below. Now by the same argument as in Case II-b (using Remark 5.5 and the assumptions from Case II), we have

$$\sigma \in \mathrm{Facet}_-(\rho) \subset \mu^{-1}(-\infty),$$

and $F_\sigma(\mathbf{b}_i - \mathbf{d}), F_\sigma(\mathbf{b}_i - \mathbf{d}_1) \geqslant M$. Therefore $\mathbf{b}_i - \mathbf{d}, \mathbf{b}_i - \mathbf{d}_1 \notin \mathbf{b}_j + \mathbb{Z}(A \cap \tau_j)$ by the definition of M (5). We have thus proved the claim (24). Hence by Lemma 3.6 we have proved (22).

We have examined all cases, and thus completed the proof. □

Sublemma 5.7. *Let τ be a face of the cone $\mathbb{R}_{\geqslant 0}A$. Then*

$$\mathbb{R}\tau = \bigcap_{\sigma \succcurlyeq \tau,\, \sigma \ \text{facet}} \mathbb{R}\sigma. \tag{25}$$

Proof. The inclusion '$\subset$' is trivial. Let $\mathbf{x}$ belong to the right-hand side of (25). Let $\sigma' \not\succcurlyeq \tau$. Then there exists $\mathbf{a}_{\sigma'} \in \tau \setminus \sigma'$. We have $F_{\sigma'}(\mathbf{a}_{\sigma'}) > 0$. So we can take $\mathbf{a}_{\sigma'}$ such that $F_{\sigma'}(\mathbf{x} + \mathbf{a}_{\sigma'}) > 0$. Do this for all $\sigma' \not\succcurlyeq \tau$. Thus we find $\mathbf{a} \in \tau$ such that $F_{\sigma'}(\mathbf{x} + \mathbf{a}) > 0$ for all $\sigma' \not\succcurlyeq \tau$.

For $\sigma \succcurlyeq \tau$, we have $F_\sigma(\mathbf{x} + \mathbf{a}) = F_\sigma(\mathbf{x}) = 0$.

Therefore $\mathbf{x} + \mathbf{a} \in \mathbb{R}_{\geqslant 0}A \cap \bigcap_{\sigma \succcurlyeq \tau} \sigma = \tau$. Hence $\mathbf{x} \in \mathbb{R}\tau$. □

Definition 5.8. For $\mathbf{d} \in \mathbb{Z}^d$, we define $P_{\mathbf{d}} \in \mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_d]$ by

$$P_{\mathbf{d}}(\theta) := \prod_{\sigma \in \mathcal{F}} \prod_{m \in F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(\mathbf{d}) + F_\sigma(\mathbb{N}A)]} \bigl(F_\sigma(\theta) - m\bigr). \tag{26}$$

Lemma 5.9. $\mathbb{I}(\Omega(\mathbf{d})) \subset \langle P_{\mathbf{d}} \rangle$.

Proof. Let σ be a facet. We aim to show that

$$\begin{aligned} &\bigl\{F_\sigma^{-1}(m)\colon\ m \in F_\sigma(\mathbb{N}A) \setminus \bigl[-F_\sigma(\mathbf{d}) + F_\sigma(\mathbb{N}A)\bigr],\ m \geqslant -F_\sigma(\mathbf{d})\bigr\} \\ &\quad \subseteq \bigl\{\mathbf{b}_i - \mathbf{d} + \mathbb{C}(A \cap \tau_i)\colon\ \mathbf{b}_i - \mathbf{d} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)\bigr\}, \end{aligned}$$

after which the result will follow from Proposition 5.1.

To verify the inclusion, take

$$\mathbf{x} \in \bigl\{F_\sigma^{-1}(m)\colon\ m \in F_\sigma(\mathbb{N}A) \setminus \bigl[-F_\sigma(\mathbf{d}) + F_\sigma(\mathbb{N}A)\bigr],\ m \geqslant -F_\sigma(\mathbf{d})\bigr\}.$$

Then because $m + F_\sigma(\mathbf{d}) \geqslant 0$, $F_\sigma(\mathbf{x} + \mathbf{d}) = m + F_\sigma(\mathbf{d}) \in F_\sigma(\mathrm{Sat}(\mathbb{N}A))$, where $\mathrm{Sat}(\mathbb{N}A)$ is the saturation of the semigroup $\mathbb{N}A$ (because we assumed $\mathbb{Z}A = \mathbb{Z}^d$, we can think of $\mathrm{Sat}(\mathbb{N}A)$ as $\mathbb{R}_{\geqslant 0}A \cap \mathbb{Z}^d$). In addition, we have $F_\sigma(\mathbf{x}) = m \in F_\sigma(\mathbb{N}A)$. Hence there exists a $\mathbf{t}_1 \in \mathbb{C}(A \cap \sigma)$ such that $\mathbf{x} + \mathbf{d} + \mathbf{t}_1 \in \mathrm{Sat}(\mathbb{N}A)$, and $\mathbf{x} + \mathbf{t}_1 \in \mathbb{N}A$. However, $\mathbf{x} + \mathbf{d} + \mathbf{t}_1 \notin \mathbb{N}A$ since $F_\sigma(\mathbf{x} + \mathbf{d} + \mathbf{t}_1) = m + F_\sigma(\mathbf{d}) \notin F_\sigma(\mathbb{N}A)$ by the definition of $\mathbf{x}$.

So we have

$$\mathbf{x}+\mathbf{d}+\mathbf{t}_1 \in \mathrm{Sat}(\mathbb{N}A) \setminus \mathbb{N}A = \mathrm{Holes}(A) = \coprod \mathbf{b}_i + \mathbb{N}(A \cap \tau_i).$$

Moreover,

$$\mathbf{x}+\mathbf{d}+\mathbf{t}_1 + \mathbb{N}(A \cap \sigma) \subset \bigcup \mathbf{b}_i + \mathbb{Z}(A \cap \tau_i),$$

and in fact the left-hand side must be contained in a *single* factor on the right-hand side. So there exists a $\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)$ with

$$\mathbf{x}+\mathbf{d}+\mathbf{t}_1 + \mathbb{N}(A \cap \sigma) \subset \mathbf{b}_i + \mathbb{Z}(A \cap \tau_i).$$

It follows that $\sigma = \tau_i$ and

$$\mathbf{x}+\mathbf{d}+\mathbf{t}_1 \in \mathbf{b}_i + \mathbb{Z}(A \cap \tau_i).$$

Now there exists a $\mathbf{t}_2 \in \mathbb{Z}(A \cap \tau_i)$ with $\mathbf{x}+\mathbf{d}+\mathbf{t}_1 = \mathbf{b}_i + \mathbf{t}_2$. Solving for $\mathbf{x}$ gives: $\mathbf{x} = \mathbf{b}_i - \mathbf{d} + (\mathbf{t}_2 - \mathbf{t}_1)$, where $\mathbf{t}_2 - \mathbf{t}_1 \in \mathbb{C}(A \cap \tau_i)$ (this last comment comes from the definition of $\mathbf{t}_1$ and the fact that $\sigma = \tau_i$). Thus $\mathbf{x} \in \mathbf{b}_i - \mathbf{d} + \mathbb{C}(A \cap \tau_i)$.

It remains to show that $\mathbf{b}_i - \mathbf{d} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$. Recall that $\mathbf{x}+\mathbf{t}_1 \in \mathbb{N}A$. Thus

$$\mathbf{b}_i - \mathbf{d} = (\mathbf{x}+\mathbf{t}_1) - \mathbf{t}_2 \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i). \qquad \square$$

We use the following lemma in Section 6.

Lemma 5.10. *Let* $\mathbf{d}_1 \in S_\mu$ *and* $\rho \subset F_{\mu,\mathbb{R}}$. *Then*

$$t^{\mathbf{d}_1+\mathbf{d}_\rho} P_{\mathbf{d}_1+\mathbf{d}_\rho}(\theta) = t^{\mathbf{d}_1} P_{\mathbf{d}_1}(\theta) \cdot t^{\mathbf{d}_\rho} P_{\mathbf{d}_\rho}(\theta).$$

Proof. We have $t^{\mathbf{d}_1} P_{\mathbf{d}_1}(\theta) \cdot t^{\mathbf{d}_\rho} P_{\mathbf{d}_\rho}(\theta) = t^{\mathbf{d}_1+\mathbf{d}_\rho} P_{\mathbf{d}_1}(\theta + \mathbf{d}_\rho) \cdot P_{\mathbf{d}_\rho}(\theta)$ (because $\theta_i = t_i \partial_i$) and

$$P_{\mathbf{d}_1+\mathbf{d}_\rho}(\theta) = \prod_{\sigma \in \mathcal{F}} \prod_{m \in F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(\mathbf{d}_1+\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)]} \big(F_\sigma(\theta) - m\big),$$

$$P_{\mathbf{d}_1}(\theta + \mathbf{d}_\rho) = \prod_{\sigma \in \mathcal{F}} \prod_{m \in [-F_\sigma(\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)] \setminus [-F_\sigma(\mathbf{d}_1+\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)]} \big(F_\sigma(\theta) - m\big),$$

$$P_{\mathbf{d}_\rho}(\theta) = \prod_{\sigma \in \mathcal{F}} \prod_{m \in F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)]} \big(F_\sigma(\theta) - m\big).$$

For $\sigma \in \mathrm{Facet}_0(\rho)$, the polynomial $P_{\mathbf{d}_\rho}(\theta)$ does not have an F_σ-factor, and the polynomials $P_{\mathbf{d}_1+\mathbf{d}_\rho}(\theta)$ and $P_{\mathbf{d}_1}(\theta + \mathbf{d}_\rho)$ have the same F_σ-factors. If $\sigma \in \mathrm{Facet}_+(\rho)$, then $\sigma \in \mu^{-1}(+\infty)$. Hence none of the above three polynomials has F_σ-factors for $\sigma \in \mathrm{Facet}_+(\rho)$, since we took $\mathbf{d}_\rho$ so that $F_\sigma(\mathbf{d}_\rho) \geqslant M$ for such a σ. Suppose that $\sigma \in \mathrm{Facet}_-(\rho)$. Then

$$
\begin{aligned}
&F_\sigma(\mathbb{N}A) \setminus \bigl[-F_\sigma(\mathbf{d}_1+\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr] \\
&\quad = \bigl(\bigl[-F_\sigma(\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr]\big\backslash\bigl[-F_\sigma(\mathbf{d}_1+\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr]\bigr) \\
&\qquad \coprod \bigl(F_\sigma(\mathbb{N}A) \setminus \bigl[-F_\sigma(\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr]\bigr),
\end{aligned}
$$

since

$$
-F_\sigma(\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A) \subset F_\sigma(\mathbb{N}A)
$$

and

$$
\bigl[-F_\sigma(\mathbf{d}_1+\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr] \subset \bigl[-F_\sigma(\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)\bigr]
$$

thanks to (6) and the properties $F_\sigma(\mathbf{d}_\rho), F_\sigma(\mathbf{d}_1) \leqslant -M$. $\square$

Lemma 5.11. *Let* $\mathbf{d}_1 \in S_\mu$, $\rho \subset F_{\mu,\mathbb{R}}$, *and* $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_\rho$. *Then*

$$
\mathbb{I}\bigl(\Omega(\mathbf{d})\bigr) = \mathbb{I}(X) \cap \langle P_{\mathbf{d}_\rho}\rangle = \mathbb{I}(X)\cdot\langle P_{\mathbf{d}_\rho}\rangle,
$$

where $X = (-\mathbf{d}_\rho + \mathbb{N}A) \setminus (-\mathbf{d} + \mathbb{N}A)$ *and, as in* (26),

$$
P_{\mathbf{d}_\rho}(\theta) = \prod_{\sigma\in\mathrm{Facet}_-(\rho)} \prod_{m\in F_\sigma(\mathbb{N}A)\setminus[F_\sigma(-\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)]} \bigl(F_\sigma(\theta)-m\bigr).
$$

Proof. First note that $F_\sigma(\mathbb{N}A) \setminus [F_\sigma(\mathbb{N}A) - F_\sigma(\mathbf{d}_\rho)] = \emptyset$ for $\sigma \in \mathrm{Facet}_+(\rho) \cup \mathrm{Facet}_0(\rho)$, since $F_\sigma(\mathbf{d}_\rho) \geqslant M$ for $\sigma \in \mathrm{Facet}_+(\rho)$. This justifies the expression for $P_{\mathbf{d}_\rho}$.

We have

$$
F_\sigma(\mathbb{N}A) + F_\sigma(\mathbf{d}_1) \subset F_\sigma(\mathbb{N}A) + F_\sigma(\mathbf{d}) \tag{27}
$$

for facets σ with $F_\sigma(\mathbf{d}_1) < 0$. Indeed, for such σ, we have $F_\sigma(\mathbf{d}_\rho) \leqslant 0$, and hence $F_\sigma(\mathbf{d}_\rho) = 0$ or $F_\sigma(\mathbf{d}_\rho) \leqslant -M$. When $F_\sigma(\mathbf{d}_\rho) = 0$, the inclusion (27) trivially holds with equality. When $F_\sigma(\mathbf{d}_\rho) \leqslant -M$, we have $\mathbb{N} \subset F_\sigma(\mathbb{N}A) + F_\sigma(\mathbf{d}_\rho)$, and hence $F_\sigma(\mathbb{N}A) + F_\sigma(\mathbf{d}_1) \subset \mathbb{N} + F_\sigma(\mathbf{d}_1) \subset F_\sigma(\mathbb{N}A) + F_\sigma(\mathbf{d})$. Thus from (14), we obtain

$$
\mathrm{ZC}\bigl((\mathbf{d}+\mathbb{N}A)\setminus\mathbb{R}_{\geqslant 0}A\bigr) = \mathrm{ZC}\bigl((\mathbf{d}_1+\mathbb{N}A)\setminus\mathbb{R}_{\geqslant 0}A\bigr) \cup \bigcup_{\sigma\in\mathrm{Facet}_-(\rho)} \bigcup_{m\in J_\sigma} F_\sigma^{-1}(m), \tag{28}
$$

where

$$
\begin{aligned}
J_\sigma &= \bigl\{m<0\colon m \in \bigl[F_\sigma(\mathbb{N}A)+F_\sigma(\mathbf{d})\bigr]\big\backslash\bigl[F_\sigma(\mathbb{N}A)+F_\sigma(\mathbf{d}_1)\bigr]\bigr\} \\
&= \bigl\{m<0\colon m \in F_\sigma(\mathbf{d}) + \bigl(F_\sigma(\mathbb{N}A)\setminus\bigl[F_\sigma(\mathbb{N}A)-F_\sigma(\mathbf{d}_\rho)\bigr]\bigr)\bigr\} \\
&= F_\sigma(\mathbf{d}) + \bigl(F_\sigma(\mathbb{N}A)\setminus\bigl[F_\sigma(\mathbb{N}A)-F_\sigma(\mathbf{d}_\rho)\bigr]\bigr).
\end{aligned} \tag{29}
$$

Note that the last equation holds, since $F_\sigma(\mathbf{d}_1) \leqslant -M$ and $m \notin F_\sigma(\mathbf{d}) + [F_\sigma(\mathbb{N}A) - F_\sigma(\mathbf{d}_\rho)] = F_\sigma(\mathbf{d}_1) + F_\sigma(\mathbb{N}A)$ imply $m \notin \mathbb{N}$. Then the first equation of the lemma follows from Eq. (11) and Lemma 5.6,

$$\begin{aligned}\mathrm{ZC}\bigl(\Omega(\mathbf{d})\bigr) &= -\mathbf{d} + \mathrm{ZC}\bigl((\mathbf{d}+\mathbb{N}A)\setminus\mathbb{N}A\bigr)\\ &= -\mathbf{d} + \Bigl[\mathrm{ZC}(\mathbf{d}+\mathbb{N}A\setminus\mathbb{R}_{\geqslant 0}A)\\ &\qquad \cup \bigcup_i \mathrm{ZC}\bigl((\mathbf{d}+\mathbb{N}A)\cap\bigl(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)\bigr)\bigr)\Bigr].\end{aligned}$$

This last equality interprets $-\mathbf{d} + \mathrm{ZC}((\mathbf{d}+\mathbb{N}A)\setminus\mathbb{N}A)$ as the Zariski-closure of the elements of $\mathbb{N}A$ that are shifted out of $\mathbb{N}A$ by adding $\mathbf{d}$. This can happen in two ways: the element can be shifted out of the cone $\mathbb{R}_{\geqslant 0}A$ entirely or the element can be shifted into the Zariski-closure of the holes, $\mathrm{ZC}((\mathbf{d}+\mathbb{N}A)\cap(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)))$. Now, using (28) and Lemma 5.6,

$$\begin{aligned}\mathrm{ZC}\bigl(\Omega(\mathbf{d})\bigr) &= -\mathbf{d} + \Bigl[\mathrm{ZC}(\mathbf{d}_1+\mathbb{N}A\setminus\mathbb{R}_{\geqslant 0}A) \cup \Bigl\{\bigcup_{\sigma\in\mathrm{Facet}_-(\rho)}\ \bigcup_{m\in J_\sigma} F_\sigma^{-1}(m)\Bigr\}\\ &\qquad \cup \bigcup_i \mathrm{ZC}\bigl((\mathbf{d}_1+\mathbb{N}A)\cap\bigl(\mathbf{b}_i+\mathbb{N}(A\cap\tau_i)\bigr)\bigr)\Bigr]\\ &= -\mathbf{d} + \Bigl[\mathrm{ZC}\bigl((\mathbf{d}_1+\mathbb{N}A)\setminus\mathbb{N}A\bigr) \cup \Bigl\{\bigcup_{\sigma\in\mathrm{Facet}_-(\rho)}\ \bigcup_{m\in J_\sigma} F_\sigma^{-1}(m)\Bigr\}\Bigr] \quad \text{(by (4))}\\ &= \bigl\{-\mathbf{d} + \mathrm{ZC}(\mathbf{d}_1+\mathbb{N}A\setminus\mathbb{N}A)\bigr\} \cup \Bigl\{\bigcup_{\sigma\in\mathrm{Facet}_-(\rho)}\ \bigcup_{m\in J_\sigma} -\mathbf{d} + F_\sigma^{-1}(m)\Bigr\}.\end{aligned}$$

Now by (29),

$$\begin{aligned}\mathrm{ZC}\bigl(\Omega(\mathbf{d})\bigr) &= \mathbb{V}\bigl(\mathbb{I}(X)\bigr) \cup \Bigl\{\bigcup_{\sigma\in\mathrm{Facet}_-(\rho)}\ \bigcup_{m\in F_\sigma(\mathbf{d})+F_\sigma(\mathbb{N}A)\setminus[F_\sigma(\mathbb{N}A)-F_\sigma(\mathbf{d}_\rho)]} -\mathbf{d} + F_\sigma^{-1}(m)\Bigr\}\\ &= \mathbb{V}\bigl(\mathbb{I}(X)\bigr) \cup \Bigl\{\bigcup_{\sigma\in\mathrm{Facet}_-(\rho)}\ \bigcup_{m\in F_\sigma(\mathbb{N}A)\setminus[F_\sigma(\mathbb{N}A)-F_\sigma(\mathbf{d}_\rho)]} F_\sigma^{-1}(m)\Bigr\}\\ &= \mathbb{V}\bigl(\mathbb{I}(X)\bigr) \cup \mathbb{V}(P_{\mathbf{d}_\rho}).\end{aligned}$$

To see that $\mathbb{I}(X)\cap\langle P_{\mathbf{d}_\rho}\rangle = \mathbb{I}(X)\cdot\langle P_{\mathbf{d}_\rho}\rangle$, it is enough to show that $X\cap\mathbb{V}(P_{\mathbf{d}_\rho}) = \emptyset$ since then $fP_{\mathbf{d}_\rho}\in\mathbb{I}(X)$ implies $f\in\mathbb{I}(X)$. But if $\sigma\in\mathrm{Facet}_-(\rho)$ and $m\in F_\sigma(\mathbb{N}A)\setminus[F_\sigma(\mathbb{N}A) - F_\sigma(\mathbf{d}_\rho)]$, then

$$F_\sigma^{-1}(m)\cap X = F_\sigma^{-1}(m)\cap\bigl((-\mathbf{d}_\rho+\mathbb{N}A)\setminus(-\mathbf{d}+\mathbb{N}A)\bigr).$$

But $m = F_\sigma(F_\sigma^{-1}(m)) \notin F_\sigma(\mathbb{N}A) - F_\sigma(\mathbf{d}_\rho)$, so $F_\sigma^{-1}(m)\cap X = \emptyset$. It follows that $X\cap\mathbb{V}(P_{\mathbf{d}_\rho}) = \emptyset$ so the second equation of the lemma holds. □

Lemma 5.12. *We have*

$$\mathbb{I}(Y) = \mathbb{I}\big(\Omega(\mathbf{d}_\rho)\big) = \langle P_{\mathbf{d}_\rho}\rangle \cdot \mathbb{I}\Bigg(\bigcup_{i\in I}\big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A\cap\tau_i)\big)\Bigg),$$

where

$$I = \left\{ i\colon \begin{array}{l} \mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A\cap\tau_i), \\ F_\sigma(\mathbf{b}_i) \in F_\sigma(\mathbb{N}A) \textit{ for all } \sigma \in \mathrm{Facet}_-(\rho) \textit{ containing } \tau_i \end{array} \right\}.$$

Proof. Recall that

$$Y = \mathbb{N}A \setminus (-\mathbf{d}_\rho + \mathbb{N}A) = -\mathbf{d}_\rho + \big[(\mathbf{d}_\rho + \mathbb{N}A)\setminus \mathbb{N}A\big].$$

By (14),

$$\mathrm{ZC}\big((\mathbf{d}_\rho + \mathbb{N}A)\setminus \mathbb{R}_{\geqslant 0}A\big) = \bigcup_{\sigma\in\mathrm{Facet}_-(\rho)} \bigcup_{m<0,\, m\in F_\sigma(\mathbb{N}A)+F_\sigma(\mathbf{d}_\rho)} F_\sigma^{-1}(m).$$

Note that $(\mathbf{d}_\rho + \mathbb{N}A)\cap(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)) = \emptyset$ for all τ_i contained in a facet $\sigma\in\mathrm{Facet}_+(\rho)$, since $F_\sigma(\mathbf{d}_\rho)\geqslant M$. Also note that $(\mathbf{d}_\rho + \mathbb{N}A)\cap(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)) = \emptyset$ for all τ_i satisfying $\mathbb{R}\tau_i \supset \rho$; otherwise the fact $\mathbf{d}_\rho \in \mathbb{Z}(A\cap\tau_i)$ contradicts the fact $\mathbf{b}_i \notin \mathbb{N}A + \mathbb{Z}(A\cap\tau_i)$. Recall from (16) that

$$(\mathbf{d}_\rho + \mathbb{N}A)\cap\big(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)\big) \neq \emptyset \iff \mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A\cap\tau_i).$$

If this is the case, then (15) implies that $\mathrm{ZC}((\mathbf{d}_\rho + \mathbb{N}A)\cap(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)))$ equals $\mathbf{b}_i + \mathbb{C}(A\cap\tau_i)$.

Hence, we obtain

$$\begin{aligned}\mathrm{ZC}(Y) = {} & \bigcup_{\sigma\in\mathrm{Facet}_-(\rho)} \bigcup_{m<-F_\sigma(\mathbf{d}_\rho),\, m\in F_\sigma(\mathbb{N}A)} F_\sigma^{-1}(m) \\ & \cup \bigcup_{\mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A\cap\tau_i)} \big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A\cap\tau_i)\big).\end{aligned}$$

Next, we claim that

$$\begin{aligned}& \bigcup_{\mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A\cap\tau_i)} \big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A\cap\tau_i)\big) \\ & = \bigcup_{\sigma\in\mathrm{Facet}_-(\rho)} \bigcup_{m\geqslant -F_\sigma(\mathbf{d}_\rho),\, m\in F_\sigma(\mathbb{N}A)\setminus[-F_\sigma(\mathbf{d}_\rho)+F_\sigma(\mathbb{N}A)]} F_\sigma^{-1}(m) \\ & \quad \cup \bigcup_{i\in I}\big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A\cap\tau_i)\big). \end{aligned} \tag{30}$$

To prove '$\subset$', let $\mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$ and $i \notin I$. Then there exists $\sigma \in \mathrm{Facet}_-(\rho)$ containing τ_i such that $F_\sigma(\mathbf{b}_i) \notin F_\sigma(\mathbb{N}A)$. We see that $m := F_\sigma(\mathbf{b}_i - \mathbf{d}_\rho) \in F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)]$ and $\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i) \subset F_\sigma^{-1}(m)$. To prove '$\supset$', let $\sigma \in \mathrm{Facet}_-(\rho)$, $m \geqslant -F_\sigma(\mathbf{d}_\rho)$, and $m \in F_\sigma(\mathbb{N}A) \setminus [-F_\sigma(\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)]$. Then $m + F_\sigma(\mathbf{d}_\rho) \in \mathbb{N} \setminus F_\sigma(\mathbb{N}A)$. Hence

$$F_\sigma^{-1}\big(m + F_\sigma(\mathbf{d}_\rho)\big) \cap \mathbb{Z}^d = \bigcup_{F_\sigma(\mathbf{b}_i)=m+F_\sigma(\mathbf{d}_\rho),\ \tau_i=\sigma} \big(\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)\big),$$

or equivalently

$$F_\sigma^{-1}(m) \cap \mathbb{Z}^d = \bigcup_{m=F_\sigma(\mathbf{b}_i-\mathbf{d}_\rho),\ \tau_i=\sigma} \big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{Z}(A \cap \tau_i)\big).$$

Since $m \in F_\sigma(\mathbb{N}A)$, there exists $\mathbf{a} \in \mathbb{N}A$ such that $\mathbf{a} + \mathbb{Z}(A \cap \sigma) \subset F_\sigma^{-1}(m) \cap \mathbb{Z}^d$. Hence there exists i such that $\mathbf{b}_i - \mathbf{d}_\rho \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$ with $m = F_\sigma(\mathbf{b}_i - \mathbf{d}_\rho)$ and $\tau_i = \sigma$. For such i, $F_\sigma^{-1}(m) = \mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i)$. This completes the proof of equality (30).

Hence $\mathbb{I}(Y) = \langle P_{\mathbf{d}_\rho} \rangle \cap \mathbb{I}\big(\bigcup_{i \in I}(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i))\big)$. Here the roots of the F_σ-factor of $P_{\mathbf{d}_\rho}$ do not belong to $-F_\sigma(\mathbf{d}_\rho) + F_\sigma(\mathbb{N}A)$ whereas those of the generators of $\mathbb{I}\big(\bigcup_{i \in I}(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i))\big)$ do. Therefore, we conclude that $\mathbb{V}(P_{\mathbf{d}_\rho}) \cap \bigcup_{i \in I}(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i)) = \emptyset$ and the assertion follows. $\square$

The following corollary is immediate from Lemmas 5.11 and 5.12.

Corollary 5.13. *Let $\mathbf{d}_1 \in S_\mu$, $\rho \subset F_{\mu,\mathbb{R}}$, and $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_\rho$. Then the deficiency ideal for the pair $\mathbf{d}_1$, $\mathbf{d}_\rho$ equals*

$$\mathbb{I}\bigg(\bigcup_{i \in I}\big(\mathbf{b}_i - \mathbf{d}_\rho + \mathbb{C}(A \cap \tau_i)\big)\bigg).$$

We are now ready to prove that all $D(R_A)$ are finitely generated. In [12], we defined a *chamber* to be the closure of a connected component of $\mathbb{R}^d \setminus \bigcup_{\sigma \in \mathcal{F}}(F_\sigma = 0)$ in the Euclidean topology.

Theorem 5.14. *Let C be any chamber. Then the $\mathbb{C}$-algebra*

$$D(R_A)_C := \bigoplus_{\mathbf{a} \in C} D(R_A)_{\mathbf{a}}$$

is finitely generated. In particular, the $\mathbb{C}$-algebra $D(R_A)$ is finitely generated.

Proof. The second claim follows from the first since there are finitely many chambers C and $D(R_A) = \bigoplus_C D(R_A)_C$.

To a map μ from $\mathcal{F}$ to $\widetilde{M}$, associate the following subspaces of $D(R_A)$:

$$D(R_A)_{S_\mu} := \bigoplus_{\mathbf{a}\in S_\mu} D(R_A)_{\mathbf{a}}, \qquad D(R_A)_{F_{\mu,\mathbb{R}}} := \bigoplus_{\mathbf{a}\in F_{\mu,\mathbb{R}}} D(R_A)_{\mathbf{a}}.$$

Then $D(R_A)_{F_{\mu,\mathbb{R}}}$ is a subalgebra of $D(R_A)$, and $D(R_A)_{S_\mu}$ is a $D(R_A)_{F_{\mu,\mathbb{R}}}$-module. We claim that

$$D(R_A)_{S_\mu} \text{ is a finitely generated } D(R_A)_{F_{\mu,\mathbb{R}}}\text{-module.} \tag{31}$$

Suppose that $\mathbf{d} \in S_\mu$ and $\rho \subset F_{\mu,\mathbb{R}}$. Recall that we took $\mathbf{d}_\rho$ so that it satisfies the condition $|F_\sigma(\mathbf{d}_\rho)| \geqslant M$ for all $\sigma \in \mathrm{Facet}_-(\rho) \cup \mathrm{Facet}_+(\rho)$. From Proposition 4.6 there exists a finite set $S_{\mu,\mathrm{fin}}$ such that

$$S_\mu = \bigcup_{\mathbf{v}\in S_{\mu,\mathrm{fin}}} \left(\mathbf{v} + \sum_{\rho\subset F_{\mu,\mathbb{R}}} \mathbb{N}\mathbf{d}_\rho\right).$$

Recall that we fixed a description of the holes of $\mathbb{N}A$:

$$\mathrm{Holes}(A) = \coprod_{i=1}^{m} \big(\mathbf{b}_i + \mathbb{N}(A\cap\tau_i)\big).$$

Assume that

$$\mathbf{d} \in S_\mu \Big\backslash \bigcup_{\mathbf{v}\in S_{\mu,\mathrm{fin}}} \left(\mathbf{v} + \sum_{\rho\subset F_{\mu,\mathbb{R}}} \mathbb{N}_{<m+2}\mathbf{d}_\rho\right),$$

where $\mathbb{N}_{<m+2}$ is the set of nonnegative integers less than $m+2$. Then there exists $\rho \subset F_{\mu,\mathbb{R}}$ such that $\mathbf{d} - k\mathbf{d}_\rho \in S_\mu$ for $k = 1, 2, \ldots, m+1$. Put $\mathbf{d}_\rho^{(k)} := k\mathbf{d}_\rho$ for $k = 1, 2, \ldots, m+1$. Then we have

(1) $F_\sigma(\mathbf{d}_\rho^{(1)}) \leqslant -M$ for all $\sigma \in \mathrm{Facet}_-(\rho)$,
(2) $F_\sigma(\mathbf{d}_\rho^{(k+1)}) - F_\sigma(\mathbf{d}_\rho^{(k)}) \leqslant -M$ for all k and $\sigma \in \mathrm{Facet}_-(\rho)$.

When Lemmas 5.11 and 5.12 are applied to $\mathbf{d}_\rho^{(k)}$, it produces three sets, $X^{(k)}$, $Y^{(k)}$, and $I^{(k)}$, corresponding to the sets X, Y, and I in Lemmas 5.11 and 5.12. Corollary 5.13 says that for all $t = 1, 2, \ldots, m+1$,

$$\mathbb{I}\big(X^{(t)}\big)\cdot\mathbb{I}\big(Y^{(t)}\big) = \mathbb{I}\big(\Omega(\mathbf{d})\big)\cdot\mathbb{I}\left(\bigcup_{i\in I^{(t)}} \big(\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A\cap\tau_i)\big)\right). \tag{32}$$

Hence

$$\sum_{t=1}^{m+1} \mathbb{I}\big(X^{(t)}\big) \cdot \mathbb{I}\big(Y^{(t)}\big) = \mathbb{I}\big(\Omega(\mathbf{d})\big) \cdot \sum_{t=1}^{m+1} \biggl(\mathbb{I}\biggl(\bigcup_{i \in I^{(t)}} \big(\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i)\big)\biggr)\biggr).$$

To prove that

$$\sum_{t=1}^{m+1} \mathbb{I}\biggl(\bigcup_{i \in I^{(t)}} \big(\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i)\big)\biggr) = (1),$$

it is enough to show that the intersection $\bigcap_{t=1}^{m+1} \bigcup_{i \in I^{(t)}} (\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i))$ is empty, since it equals

$$\mathbb{V}\biggl(\sum_{t=1}^{m+1} \mathbb{I}\biggl(\bigcup_{i \in I^{(t)}} \big(\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i)\big)\biggr)\biggr).$$

Suppose that the intersection $\bigcap_{t=1}^{m+1} \bigcup_{i \in I^{(t)}} (\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i))$ is nonempty; we aim for a contradiction. By the pigeon-hole principle, there exists an index i $(1 \leqslant i \leqslant m)$ and two numbers t and t' between 1 and $m+1$ such that

$$\big[\mathbf{b}_i - \mathbf{d}_\rho^{(t)} + \mathbb{C}(A \cap \tau_i)\big] \cap \big[\mathbf{b}_i - \mathbf{d}_\rho^{(t')} + \mathbb{C}(A \cap \tau_i)\big] \neq \emptyset.$$

Then

$$\mathbf{d}_\rho^{(t)} - \mathbf{d}_\rho^{(t')} \in \mathbb{C}(A \cap \tau_i) \cap \mathbb{Z}^d.$$

But this last element is just a multiple of $\mathbf{e}_\rho$ so $\rho \subset \mathbb{R}\tau_i$. Then $\mathbf{d}_\rho \in \mathbb{Z}(A \cap \tau_i)$ by (17). Now $\mathbf{b}_i - \mathbf{d}_\rho^{(t)} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$ because that is how we defined the set I in Lemma 5.12. But together with $\mathbf{d}_\rho \in \mathbb{Z}(A \cap \tau_i)$, this gives $\mathbf{b}_i \in \mathbb{N}A + \mathbb{Z}(A \cap \tau_i)$. This cannot be the case, because $\mathbf{b}_i$ was a “hole.” So we get a contradiction. Thus, the intersection is empty and

$$\sum_{t=1}^{m+1} \mathbb{I}\big(X^{(t)}\big) \cdot \mathbb{I}\big(Y^{(t)}\big) = \mathbb{I}\big(\Omega(\mathbf{d})\big).$$

Therefore, we obtain

$$D(R)_{\mathbf{d}} = \sum_{t=1}^{m+1} D(R)_{\mathbf{d}-\mathbf{d}_\rho^{(t)}} D(R)_{\mathbf{d}_\rho^{(t)}}.$$

The above argument shows the claim (31), more precisely, that $D(R_A)_{S_\mu}$ is generated by $\bigoplus_{\mathbf{d}} D(R_A)_{\mathbf{d}}$ with $\mathbf{d}$ running over the finite set

$$\bigcup_{\mathbf{v}\in S_{\mu,\mathrm{fin}}} \left(\mathbf{v} + \sum_{\rho\subset F_{\mu,\mathbb{R}}} \mathbb{N}_{<m+2}\mathbf{d}_\rho\right)$$

as a right $D(R_A)_{F_{\mu,\mathbb{R}}}$-module.

For any chamber C, $C \cap \mathbb{Z}^d = \bigcup_{S_\mu\subset C} S_\mu$. Moreover, $S_\mu \subset C \Rightarrow F_{\mu,\mathbb{R}} \subset C$ and $F_{\mu,\mathbb{R}} \cap \mathbb{Z}^d = \bigcup_{S_{\mu'}\subset F_{\mu,\mathbb{R}}} S_{\mu'}$. Hence, the above argument also shows that the $\mathbb{C}$-algebra $D(R_A)_C$ is finitely generated. Thus, we have proved the theorem. □

6. Finite generation of Gr $D(R_A)$ for scored semigroups

In this section we prove that if $\mathbb{N}A$ is scored, then $\operatorname{Gr} D(R_A)$ is finitely generated. Together with [12, Theorem 3.2.12], this completes the proof of Theorem 1.1(1).

Throughout this section, we assume $\mathbb{N}A$ *to be scored*.

Proposition 6.1. $\mathbb{I}(\Omega(\mathbf{d})) = \langle P_{\mathbf{d}}\rangle$.

Proof. Since $\mathbb{N}A$ is scored, $\mathbf{a} \in \Omega(\mathbf{d})$ if and only if $F_\sigma(\mathbf{a}) \in F_\sigma(\mathbb{N}A)$ for all $\sigma \in \mathcal{F}$, and $F_{\sigma'}(\mathbf{a}) \notin -F_{\sigma'}(\mathbf{d}) + F_{\sigma'}(\mathbb{N}A)$ for some $\sigma' \in \mathcal{F}$. □

Corollary 6.2.

$$\operatorname{Gr}\big(D(R_A)\big) = \bigoplus_{\mathbf{d}\in\mathbb{Z}^d} t^{\mathbf{d}}\mathbb{C}[\bar\theta_1, \bar\theta_2, \ldots, \bar\theta_d](\overline{P}_{\mathbf{d}}),$$

where $\bar\theta_j = t_j\xi_j$ *and*

$$\overline{P}_{\mathbf{d}} = \prod_{\sigma\in\mathcal{F}} F_\sigma(\bar\theta_1, \bar\theta_2, \ldots, \bar\theta_d)^{\sharp(F_\sigma(\mathbb{N}A)\setminus(-F_\sigma(\mathbf{d})+F_\sigma(\mathbb{N}A)))}.$$

Proof. This follows immediately from Proposition 6.1, Theorem 2.1, and the definition of $P_{\mathbf{d}}$ in (26). □

Theorem 6.3. *Let C be any chamber. Then the* $\mathbb{C}$*-algebra*

$$\operatorname{Gr}\big(D(R_A)_C\big) = \operatorname{Gr}\big(D(R_A)\big)_C := \bigoplus_{\mathbf{a}\in C} \operatorname{Gr}\big(D(R_A)\big)_{\mathbf{a}}$$

is finitely generated. Moreover, the $\mathbb{C}$*-algebra* $\operatorname{Gr}(D(R_A))$ *is finitely generated.*

Proof. For each $\rho \in \mathrm{Ray}(A)$, we took $\mathbf{d}_\rho$ so that it satisfies the condition $|F_\sigma(\mathbf{d}_\rho)| \geqslant M$ for all $\sigma \in \mathrm{Facet}_-(\rho) \cup \mathrm{Facet}_+(\rho)$. For any μ, as in Section 4, there exists a finite set $S_{\mu,\mathrm{fin}}$ such that

$$S_\mu = \bigcup_{\mathbf{v} \in S_{\mu,\mathrm{fin}}} \left(\mathbf{v} + \sum_{\rho \subset F_{\mu,\mathbb{R}}} \mathbb{N}\mathbf{d}_\rho \right).$$

Assume that $\mathbf{d} \in S_\mu \setminus S_{\mu,\mathrm{fin}}$. Then there exists a ray $\rho \subset F_{\mu,\mathbb{R}}$ such that $\mathbf{d} - \mathbf{d}_\rho \in S_\mu$. By Lemma 5.10 and Corollary 6.2, we have

$$\mathrm{Gr}\big(D(R)\big)_{\mathbf{d}} = \mathrm{Gr}\big(D(R)\big)_{\mathbf{d}-\mathbf{d}_\rho} \cdot \mathrm{Gr}\big(D(R)\big)_{\mathbf{d}_\rho}.$$

Hence $\mathrm{Gr}(D(R_A))_{S_\mu}$ is generated by $\bigoplus_{\mathbf{d} \in S_{\mu,\mathrm{fin}}} D(R_A)_{\mathbf{d}}$ as a right $D(R_A)_{F_{\mu,\mathbb{R}}}$-module.

The same remark as in the final paragraph of the proof of Theorem 5.14 shows that $\mathrm{Gr}(D(R_A))$ is finitely generated. □

Corollary 6.4. *Let C be a chamber. If $\mathbb{N}A$ is scored, then*

(1) $\mathrm{Gr}(D(R_A))_C$ *and* $\mathrm{Gr}(D(R_A))$ *are Noetherian*;
(2) $D(R_A)_C$ *and* $D(R_A)$ *are left and right Noetherian.*

Proof. (1) is an immediate consequence of Hilbert's basis theorem.

(2) follows from the standard argument using induction on the order of differential operators: Let $\{I_n\}_{n=1,2,\ldots}$ be an increasing sequence of left ideals of $D(R_A)_C$. Define a filtration F of each I_n by $F_m(I_n) := D_m(R_A) \cap I_n$ and put $\mathrm{Gr}(I_n) := \bigoplus_{m=0}^\infty F_{m+1}(I_n)/F_m(I_n)$. Then $\{\mathrm{Gr}(I_n)\}$ is an increasing sequence of ideals of $\mathrm{Gr}(D(R_A))_C$. By (1), there exists N such that $\mathrm{Gr}(I_{N+k}) = \mathrm{Gr}(I_N)$ for all $k \in \mathbb{N}$. Suppose that $I_N \subsetneq I_{N+k}$. Take the smallest m such that $F_m(I_N) \subsetneq F_m(I_{N+k})$. Then $F_{m-1}(I_N) = F_{m-1}(I_{N+k})$ and $\mathrm{Gr}_m(I_N) = \mathrm{Gr}_m(I_{N+k})$ imply $F_m(I_N) = F_m(I_{N+k})$, which contradicts the choice of m.

The right Noetherian property can be proved similarly. □

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